1/24/24 EXAMPLES + PROPERTIES OF DTFTs

(OS4P 2.6-2.9, NEXT 3.0-3.2)

DTFT FOR $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

IDTFT $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

EXAMPLE

$x[n] = \alpha^n u[n], \ |\alpha| < 1 \iff X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$

$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \cdot \underbrace{\frac{1-\alpha e^{j\omega}}{1-\alpha e^{j\omega}}}_{1} = \frac{1-\alpha e^{j\omega}}{1-\underbrace{\alpha e^{j\omega}-\alpha e^{-j\omega}}_{-2\alpha\cos(\omega)}+\alpha^2}$

$= \frac{1-\alpha e^{j\omega}}{1-2\alpha\cos(\omega)+\alpha^2} = \frac{1-\alpha\cos(\omega) - j\alpha\sin(\omega)}{(1+\alpha^2) - 2\alpha\cos(\omega)}$

$\text{Re}\{X(e^{j\omega})\}$

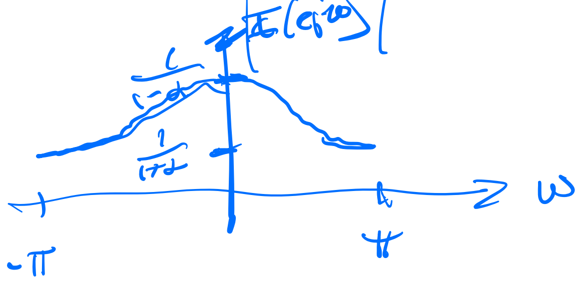
$\text{Im}[X(e^{j\omega})] = \frac{-\alpha\sin(\omega)}{(1+\alpha^2)-2\alpha\cos(\omega)}$

$|X(e^{j\omega})| = \sqrt{\frac{(1-\alpha\cos(\omega))^2 + \alpha^2\sin^2(\omega)}{(1-2\alpha\cos(\omega)+\alpha^2)^2}} = \frac{1-2\alpha\cos(\omega)+\cancel{\alpha^2\cos^2(\omega)}+\cancel{\alpha^2\sin^2(\omega)}\ \alpha^2}{\sqrt{(1-2\alpha\cos(\omega)+\alpha^2)^2}}$

$= \frac{\sqrt{1-2\alpha\cos(\omega)+\alpha^2}}{1-2\alpha\cos(\omega)+\alpha^2} = \frac{1}{\sqrt{1-2\alpha\cos(\omega)+\alpha^2}}$

FOR $\omega = 0$, $|X(e^{j\omega})| = \frac{1}{(1-2\alpha+\alpha^2)^{1/2}} = \frac{1}{1-\alpha}$

$\omega = \pi$, $|X(e^{j\omega})| = \frac{1}{1+\alpha}$

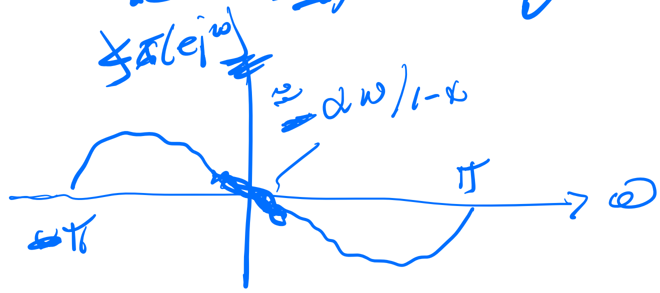


$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{Im[X(e^{j\omega})]}{Re[X(e^{j\omega})]}\right) = \tan^{-1}\left[\frac{-\alpha \sin(\omega)}{1-\alpha\cos(\omega)}\right]$$

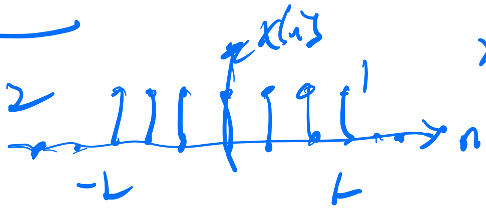
IF $|\omega| \ll 1$

$\sin(\omega) \approx \omega$

$\cos(\omega) \approx 1$



Ex 2



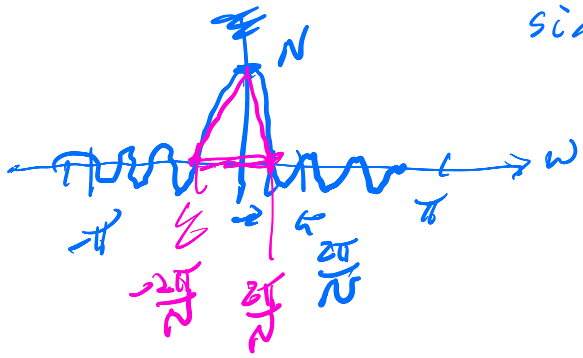
$x[n] = 1, \ |n| \le L$

$0, \ \text{ELSE}$

Let $N = 2L+1$

$$X(e^{j\omega}) = \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$\Longleftrightarrow$

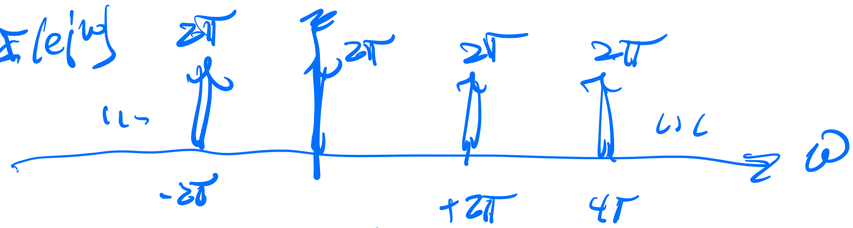


ZERO CROSSINGS @

$$\frac{\omega N}{2} = k\pi, \quad \omega = k\frac{2\pi}{N}$$

$$\text{AREA} = \frac{1}{2}\cdot\frac{4\pi}{\cancel{N}}\cancel{N} \Rightarrow 2\pi\,\delta(\omega)$$

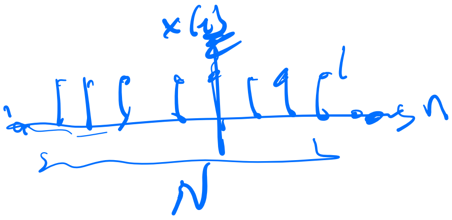
$\lim_{N\to\infty} X(e^{j\omega})$



$$1 \Longleftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\,\delta(\omega - 2\pi r)$$

EXAMPLES

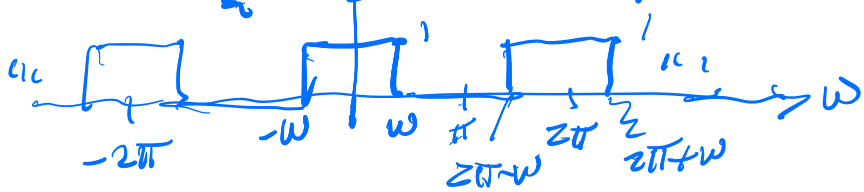
1. $x[n] = \alpha^n u[n] \Longleftrightarrow \dfrac{1}{1-\alpha e^{-j\omega}}$

2. 

$\Longleftrightarrow \dfrac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

3. $h[n] = \int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$

$\Longleftrightarrow H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le W \\ 0 & W < |\omega| \le \pi \end{cases}$



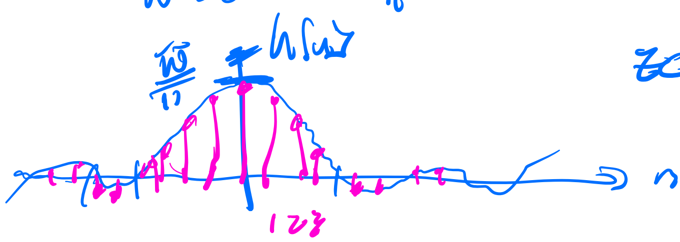
$= \frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{j\omega n}\, d\omega$

$\rightarrow \int e^{ax}\,dx = \frac{1}{a} e^{ax}$

$= \frac{1}{2\pi}\frac{1}{jn}\left[e^{j\omega n}\right]_{\omega=-W}^{W} = \frac{1}{2\pi}\frac{1}{jn}\left[e^{jWn} - e^{-jWn}\right] = \frac{\cancel{2j}\sin(Wn)}{\cancel{2}\pi j n}$

$= \frac{\sin(Wn)}{\pi n}$ $\qquad \lim_{n\to 0} = \frac{W\cos(Wn)}{\pi} = \frac{W}{\pi}$

$= \frac{W}{\pi} \cdot \frac{\sin(Wn)}{Wn}$



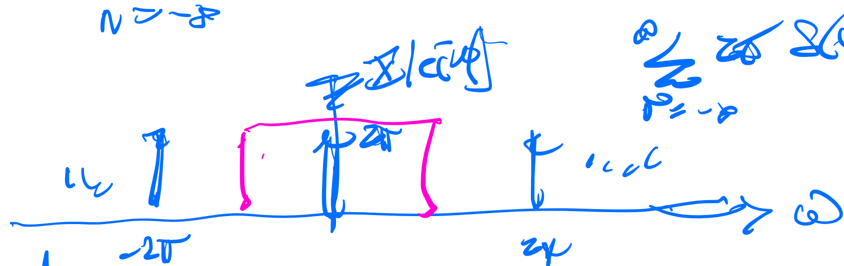
zeros $Wn = k\pi$

$n = k\pi/W$

4. $x[n] = \delta[n]$ $\qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$

5. $x[n] = 1 \Longleftrightarrow$

$\sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - 2\pi k)$



$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$

PROPERTIES

1. LINEARITY

$$a x_1[n] + b x_2[n] \Leftrightarrow a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

2. TIME SHIFT IF $x[n] \Leftrightarrow X(e^{j\omega})$

$$x[n-N] \Leftrightarrow \sum_{n=-\infty}^{\infty} x[n-N]\, e^{-j\omega n} = \sum_{\ell=-\infty}^{\infty} x[\ell]\, e^{-j\omega(\ell+N)}$$

let $\ell = n-N$

$n = \ell + N$

$$= e^{-j\omega N} \cdot \sum_{\ell=-\infty}^{\infty} x[\ell]\, e^{-j\omega \ell} = e^{-j\omega N} X(e^{j\omega})$$

3. MULT BY COMPLEX EXPONENTIAL

$$x[n] \cdot e^{j\omega_0 n} \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

4. CONVOLUTION IN TIME

$$x[n] \longrightarrow \overset{\text{LTI}}{\boxed{h[n]}} \longrightarrow y[n]$$

IF $x[n] \Leftrightarrow X(e^{j\omega})$ $y[n] = x[n]\, h[n]$

$h[n] \Leftrightarrow H(e^{j\omega})$ FIND $Y(e^{j\omega})$ IN TERMS OF $X(e^{j\omega}), H(e^{j\omega})$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \underbrace{\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}_{y[n]} e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\sum_{n=-\infty}^{\infty} h[n-k]\, e^{-j\omega n}}_{H(e^{j\omega})\, e^{-j\omega k}} = H(e^{j\omega}) \underbrace{\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}}_{X(e^{j\omega})}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

5. MULTIPLICATION ON TIME

$$x_1[n]\, x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$