9/4/24

CHANGE IN SAMPLING RATE

LO 3.0 - 3.3, NOTES 1.4

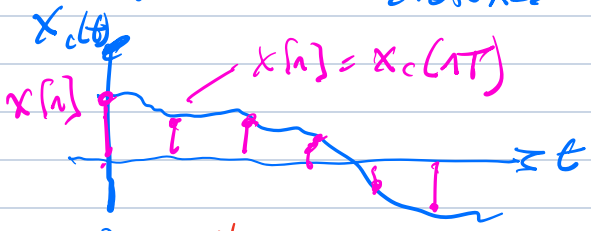
DTFT

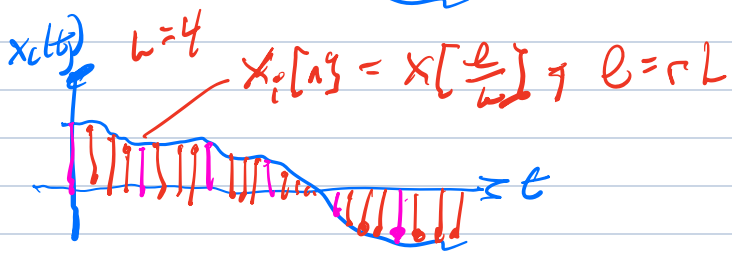
IF $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

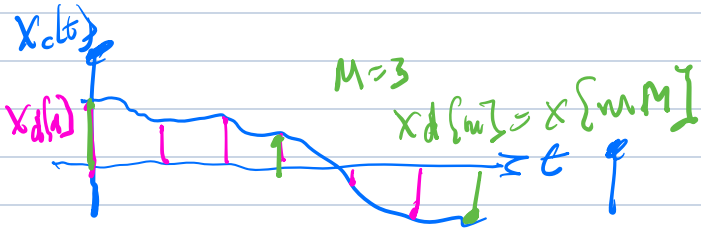
SAMPLING OF SIGNALS





UPSAMPLING (INTERPOLATION) BY $L=4$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_i[\ell]$$



DOWNSAMPLING (DECIMATION) BY $M=3$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[m]$$

INCORRECT FREQ ANALYSIS

WRONG!

$$x_d[m] = x[mM]$$

$$X_d(e^{j\omega'}) = \sum_{m=-\infty}^{\infty} x_d[m] e^{-j\omega' m} = \sum_{m=-\infty}^{\infty} x[mM] e^{-j\omega' m}$$

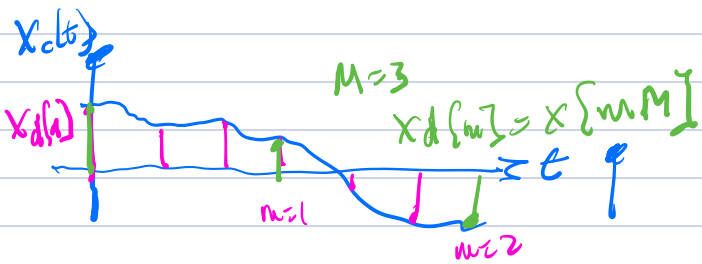
let $s = mM$, $m = s/M$

$$= \sum_{s=-\infty}^{\infty} x[s] e^{-j\omega' s/M}$$

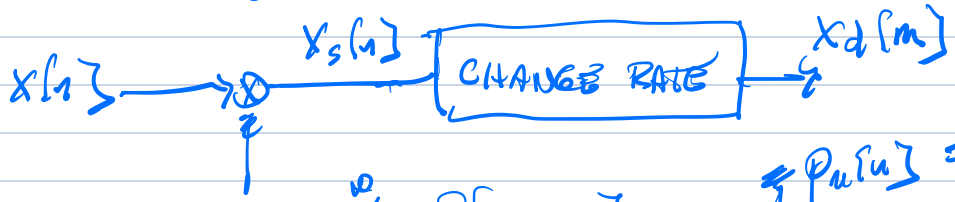
$e^{-j\frac{\omega'}{M}s}$

$$X_d(e^{j\omega'}) = X(e^{j\omega'/M})$$

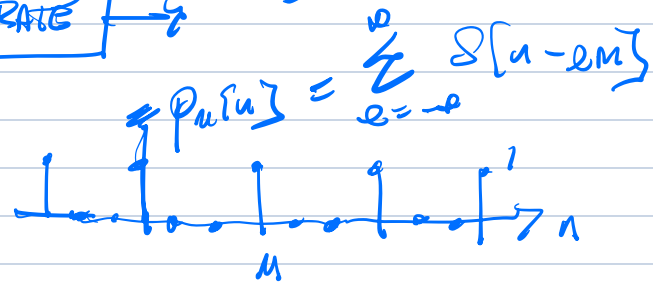
WHAT'S WRONG?

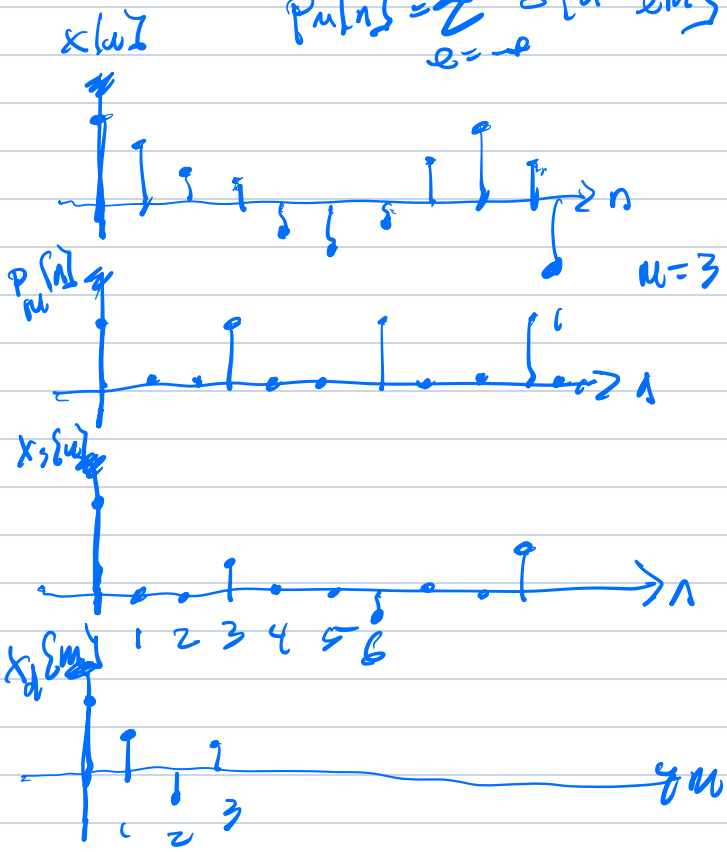


CORRECT WAY TO DECIMATE BY M



$$p_M[n] = \sum_{\ell=-\infty}^{\infty} \delta[n-\ell M]$$





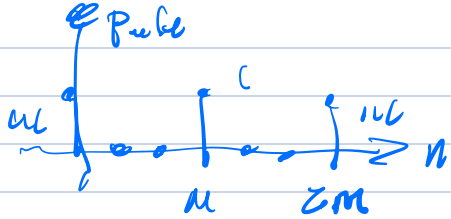
$$x_s[n] = x[n]\, p_M[n]$$

$$\Updownarrow$$

$$X_s(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \circledast P_M(e^{j\omega})$$

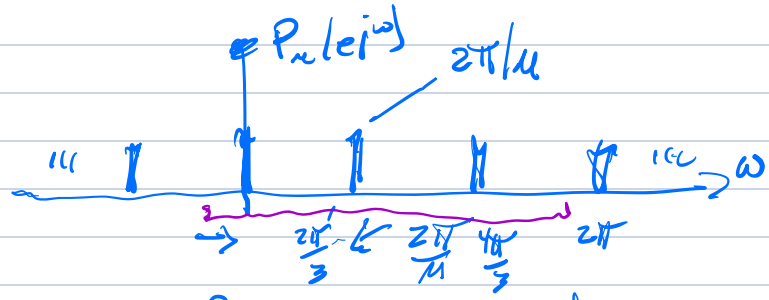
$$X_s(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, P_M(e^{j(\omega-\theta)})\, d\theta$$

$p_M[n] =$



$= 1, \; n = lM$

$0, \; \text{OTHERWISE}$



$$P_M(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{M} \delta\left(\omega - \frac{2\pi k}{M}\right)$$

$$P_M(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_{l=0}^{M-1} \frac{2\pi}{M} \delta\left(\omega - \frac{2\pi l}{M} - 2\pi r\right)$$

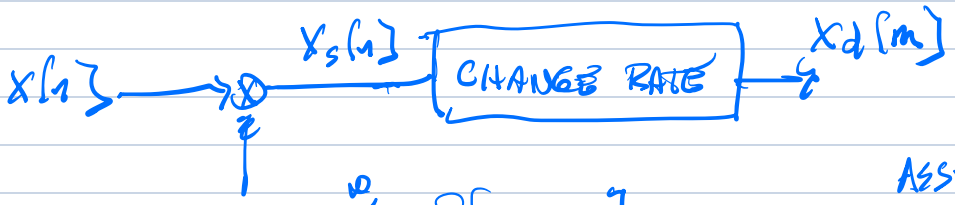
$$p_M[n] = \frac{1}{2\pi} \int_{0-\epsilon}^{2\pi-\epsilon} P_M(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{0-\epsilon}^{2\pi-\epsilon} \frac{2\pi}{M} \sum_{l=0}^{M-1} \delta\left(\omega - \frac{2\pi l}{M}\right) e^{j\omega n} d\omega$$

$$= \frac{1}{M} \sum_{l=0}^{M-1} \int_{0-\epsilon}^{2\pi-\epsilon} \delta\left(\omega - \frac{2\pi l}{M}\right) e^{j\omega n} d\omega$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

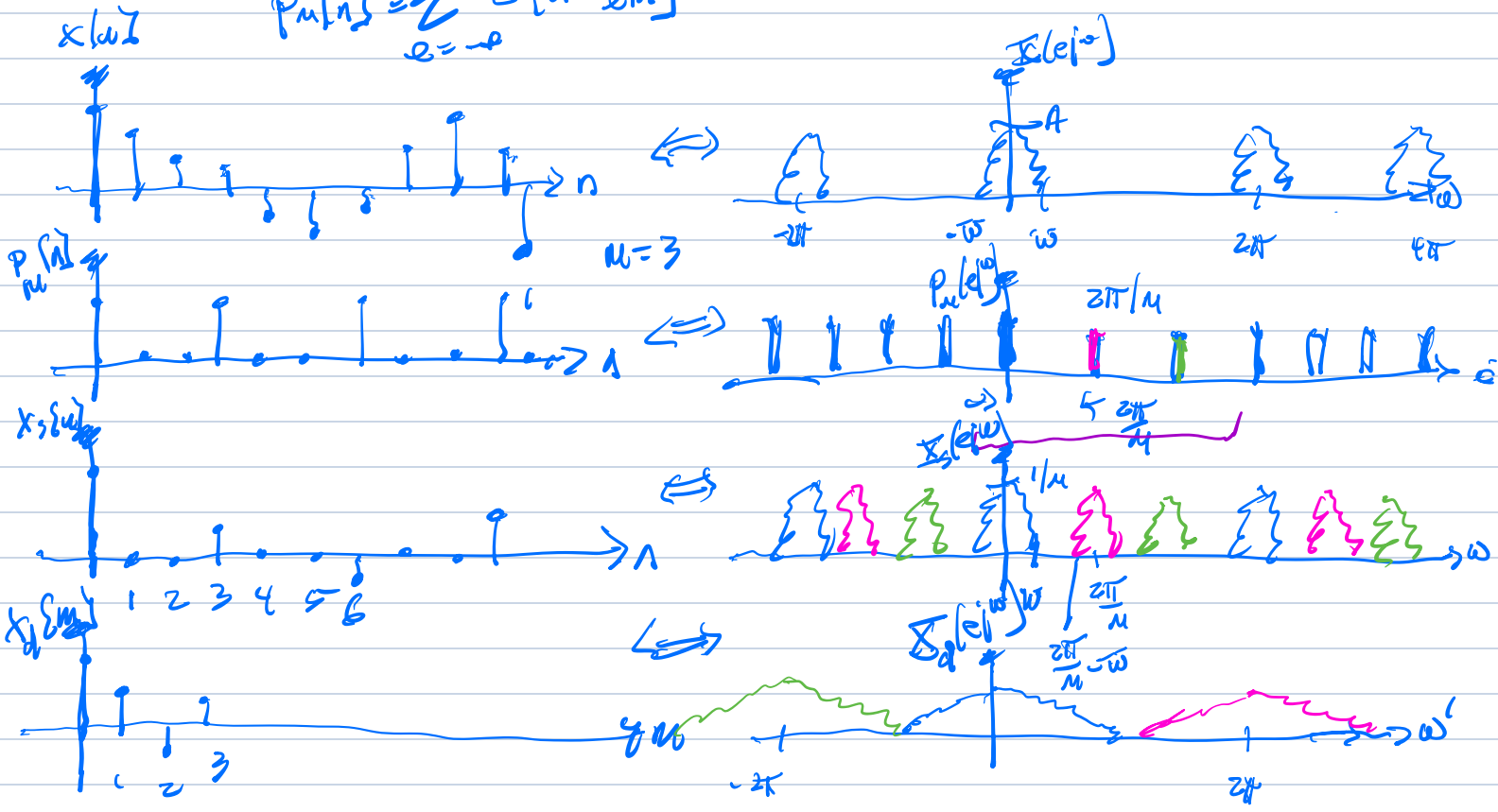
$$\frac{1}{M} \sum_{l=0}^{M-1} e^{j\frac{2\pi l}{M}} = \frac{1}{M} \frac{1 - e^{j\frac{2\pi l}{M} M}}{1 - e^{j\frac{2\pi l}{M}}}$$

$$= \begin{cases} 1, & n = rM \\ 0, & \text{ELSE} \end{cases}$$



$$p_M[n] = \sum_{\ell=-\infty}^{\infty} \delta[n-\ell M]$$

ASSUME $|X(e^{j\omega})| = 0,\ \omega < |\omega| \le \pi$



$$X_s(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \circledast P_M(e^{j\omega})$$

TO AVOID AL ALIASING REQUIRE $\pi < \frac{2\pi}{M} - \pi$

$2\pi < \frac{2\pi}{M}$, $\boxed{\pi < \frac{\pi}{M}}$

$$x_d[n] = x_s[mM]$$

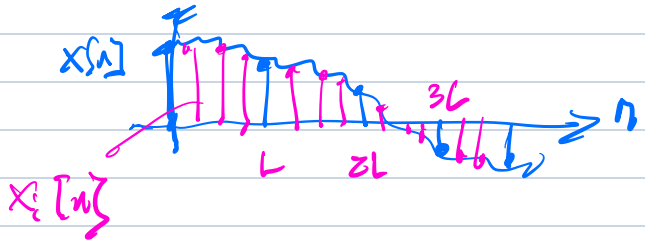
$$X_d(e^{j\omega'}) = \sum_{n=-\infty}^{\infty} x_s[n] e^{-j\omega' n} = \sum_{m=-\infty}^{\infty} x_s[mM] e^{-j\omega' m} = \sum_{s=-\infty}^{\infty} x[s] e^{-j\frac{\omega'}{M} s}$$

$n = rM \quad s = mM$

$m = \frac{s}{M}$

$$= X(e^{j\frac{\omega'}{M}})$$

INTERPOLATION BY $L = 4$



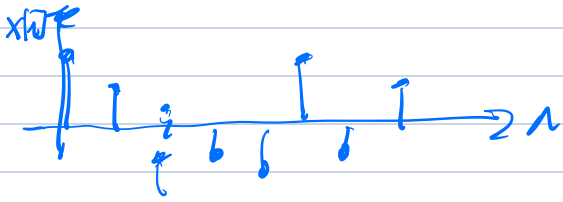
$$x_i[\ell] = x\left[\frac{\ell}{L}\right], \quad \ell = rL$$

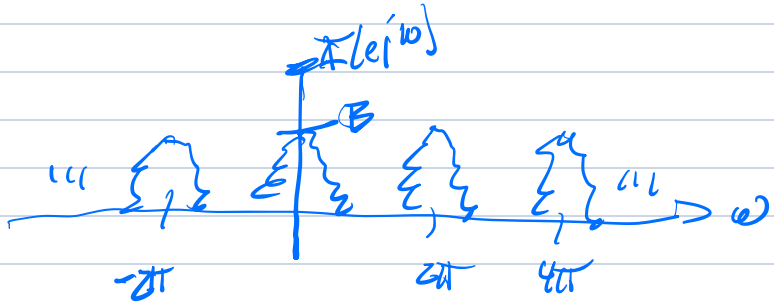
$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_i[\ell]$$

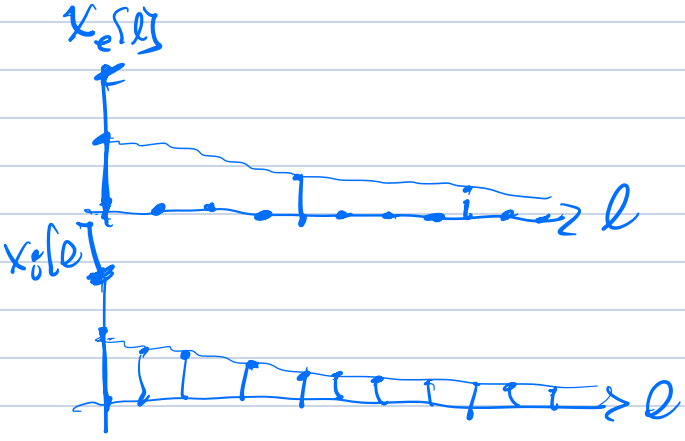
APPROACH

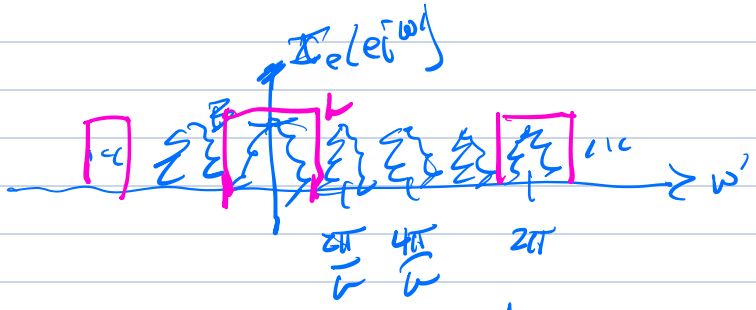
$$x[n] \longrightarrow \boxed{\text{INSERT } L-1 \text{ SAMPLES OF ZERO}} \xrightarrow{x_e[\ell]} \boxed{\text{LPF}} \longrightarrow x_i[\ell]$$

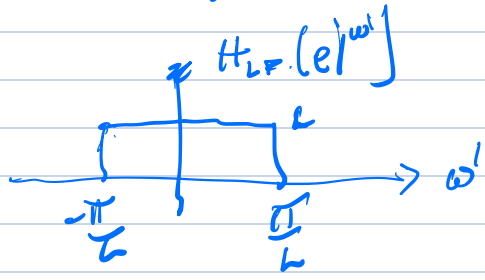
$L = 4$











$$X_e(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x_e[\ell]\, e^{-j\omega'\ell}$$

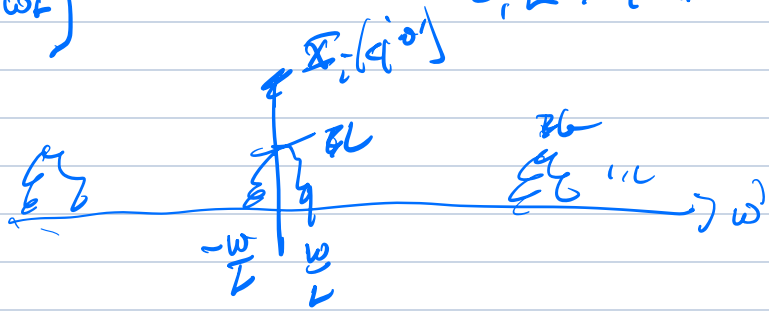
$$= \sum_{\ell=-\infty}^{\infty} x\left[\frac{\ell}{L}\right] e^{-j\omega'\ell}$$

$\ell = rL$ $\quad$ let $s = \frac{\ell}{L}$

$\ell = sL$

$$X_e(e^{j\omega'}) = \sum_{s=-\infty}^{\infty} x[s]\, e^{-j\omega' L s} = X(e^{j\omega' L})$$

$$X_i(e^{j\omega'}) = X(e^{jL\omega'})$$

$|\omega'| < \frac{\pi}{L}$

$0, \; L < |\omega'| \le \pi$