Carnegie Mellon DSP Electrical & Computer ENGINEERING

Fundamentals of Signal Processing (18-491)
Spring Semester, 2019

NOTES FOR 18- 491 LECTURES 3 and 4

Introduction to Discrete-Time Fourier Transforms (DTFTs)

I. Introduction

Last week we talked about the time domain behavior of discrete-time signals and systems. Today we will review and discuss how we characterize discrete-time signals and systems in the frequency domain using the discrete-time Fourier transform.

II. The DTFT and its inverse

Probably the easiest way to introduce the discrete-time Fourier transform (DTFT) is through its counterpart, the continuous-time Fourier transform (CTFT). As you will recall, the CTFT and its inverse can be expressed as:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}dt \tag{1}$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{2}$$

Note that we use the upper case variable Ω to indicate continuous-time angular frequency in this course. As we have discused, the first equation (which is actually the inverse CTFT) expresses the fact that a finite-energy time function $x(t)$ can be represented as a weighted linear combination of complex exponentials $e^{j\Omega t}$. The second equation (the actual CTFT) tells you how to compute the weights $X(j\Omega)$. Note that the frequencies Ω include all real numbers.

The DTFT $X(e^{j\omega})$ is computed in very similar fashion to the CTFT:

$$X(e^{j\omega n}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n} \tag{3}$$

Note that the complex exponential $e^{j\omega n}$ in Eq. 3 has the same role as the complex exponential $e^{j\Omega t}$ in Eq. 1. Now let's consider the expression $e^{j\omega n}$ for the frequencies ω_0 and $\omega_0 + 2\pi$:

$$e^{j(\omega_0 + 2\pi)n} = e^{j\omega_0 n} e^{j2\pi n} = e^{j\omega_0 n} \tag{4}$$

In other words, the discrete-time frequency variable is periodic with period 2π. As a result, we only evaluate the IDTFT computation over a frequency region that is arbitrary but of extent 2π: This periodicity occurs because the time variable *n* is always integer.

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{5}$$

where the single subscript on the integral sign indicates that integration can be performed over *any* strip of ω of extent 2π. This occurs because the entire integrand of Eq. 5 is periodic.

The validity of Eqs. (3) and (5) can be confirmed by simple substitution:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{\infty} x[l] e^{-j\omega l} e^{j\omega n} d\omega \tag{6}$$

$$= \sum_{l=-\infty}^{\infty} x[l] \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{\infty} e^{-j\omega l} e^{j\omega n} d\omega \tag{7}$$

Because *n* and *l* are always integers, the integral in Eq. *(7)* is equal to 2π when $n = l$ and zero otherwise. Hence the only nonzero term in the outer sum occurs when $l = n$, and it evaluates to $x[n]$. This confirms that Eqs. (3) and (5) are transform pairs.

III. Basic DTFT examples

The decaying exponential. We first consider the simple function

$$x[n] = \alpha^n u[n], 0 < n < 1 \tag{8}$$

Substituting directly produces

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}} \tag{9}$$

As shown in class, the real and imaginary parts can be obtained by rationalizing the denominator:

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)\left(\frac{1-\alpha e^{j\omega}}{1-\alpha e^{j\omega}}\right) = \frac{1-\alpha e^{j\omega}}{1-2\alpha\cos(\omega)+\alpha^2} = \frac{1-\alpha\cos(\omega)-j\alpha\sin(\omega)}{1-2\alpha\cos(\omega)+\alpha^2} \quad (10)$$

By comparison of the terms we obtain

$$Re[X(e^{j\omega})] = \frac{1-\alpha\cos(\omega)}{2-2\alpha\cos(\omega)+\alpha^2} \quad (11)$$

and

$$Im[X(e^{j\omega})] = \frac{-j\alpha\sin(\omega)}{2-2\alpha\cos(\omega)+\alpha^2} \quad (12)$$

Because $|Z|^2 = \sqrt{(Re[Z])^2 + (Im[Z])^2}$ and $\angle Z = \text{atan}\left(\frac{Im[Z]}{Re[Z]}\right)$ for any complex variable *Z*,

$$\left|X(e^{j\omega})\right|^2 = \left(\frac{1-\alpha\cos(\omega)}{1-2\alpha\cos(\omega)+\alpha^2}\right)^2 + \left(\frac{-\alpha\sin(\omega)}{1-2\alpha\cos(\omega)+\alpha^2}\right)^2 = \frac{1-2\alpha\cos(\omega)+\alpha^2}{(1-2\alpha\cos(\omega)+\alpha^2)^2} \quad (13)$$

Hence

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1-2\alpha\cos(\omega)+\alpha^2}} \quad (14)$$

and

$$\angle X(e^{j\omega}) = \text{atan}\left(\frac{-\alpha\sin(\omega)}{1-\alpha\cos(\omega)}\right) \quad (15)$$

We note that $Re[X(e^{j\omega})]$ is an even function of ω, $Im[X(e^{j\omega})]$ is odd, $\left|X(e^{j\omega})\right|$ is even and $\angle X(e^{j\omega})$ is odd, at least for this particular time function.

E2. The finite-duration pulse. We next consider the finite duration causal pulse,

$$x[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0, \text{ otherwise} \end{cases} \quad (16)$$

This sum is also easily obtained:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \sum_{n=0}^{N-1} (e^{-j\omega})^n = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \quad (17)$$

Please note that we used the relation for the finite sum of exponentials discussed in recition in developing the last result. This expression can be further simplified by balancing the terms in the parentheses:

$$X(e^{j\omega}) = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \left(\frac{e^{-j\omega N/2}}{e^{-j\omega/2}}\right)\left(\frac{e^{j\omega N/2}-e^{-j\omega N/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) = e^{-j\omega(N-1)/2}\left(\frac{2j\sin(N\omega/2)}{2j\sin(\omega/2)}\right) \tag{18}$$

or

$$X(e^{j\omega}) = e^{-j\omega(N-1)/2}\left(\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right) \tag{19}$$

The term on the left contributes a linear phase shift to the DTFT. The quantity inside the parentheses, $\frac{\sin(N\omega/2)}{\sin(\omega/2)}$, is sometimes referred to as the "discrete-time sinc function". It has the following properties:

- By L'Hopital's rule, $\lim_{\omega \to 0} \frac{\sin(N\omega/2)}{\sin(\omega/2)} = N$
- $\frac{\sin(N\omega/2)}{\sin(\omega/2)}$ has regularly-occurring zero crossings at $\omega = 2\pi k/N$ for all integer k
- The envelope of $\frac{\sin(N\omega/2)}{\sin(\omega/2)}$ tapers downward as $|\omega|$ increases from zero to π

IV. Some properties and additional examples of DTFTs

In this section we summarize some of the properties of DTFTs along with some additional DTFT examples. In some cases, brief proofs were provided in the lecture ... for the most part these proofs are not included in these notes in the interests of brevity.

P1. Linearity. As noted in class, the DTFT operation itself is linear.

P2. Time shift. $x[n-N] \Leftrightarrow X(e^{j\omega})e^{-j\omega N}$

P3. Multiplication by a complex exponential. $x[n]e^{j\omega_0 n} \Leftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$

E3. DTFT of an impulse. $\delta[n] \Leftrightarrow \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1$ for all *n*.

E4. DTFT of a constant. In similar fashion, working from the right side we obtain

$$1 \Leftrightarrow 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - 2\pi r) \tag{20}$$

While this notation is cumbersome, it merely expresses the fact that a constant in time has a DTFT that is a delta function in frequency at $\omega = 0$ and that this function repreats periodically in frequency with period

2π.

E5. DTFT of a complex exponential. Using Property P3, we can now easily obtain

$$e^{j\omega_0 n} \Leftrightarrow 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi r). \tag{21}$$

This is simply a shifted delta function that is repeated periodically.

E6. DTFT of a cosine. From Euler's representation of trig functions we obtain

$$\cos(\omega_0 n) = \frac{e^{j\omega_0} + e^{-j\omega_0 n}}{2} \Leftrightarrow \pi \sum_{r=-\infty}^{\infty} (\delta(\omega - \omega_0 - 2\pi r) + \delta(\omega + \omega_0 - 2\pi r)) \tag{22}$$

This is simply a pair of delta functions of area π occuring at $\omega = \pm\omega_0$, repeating periodically with period 2π.

P4. Time reversal. $x[-n] \Leftrightarrow X(e^{-j\omega})$

We note that if a time function is *even*, $x[n] = x[-n] \Leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$

Similarly, if a time function is *odd,* $x[n] = -x[-n] \Leftrightarrow X(e^{j\omega}) = -X'(e^{-j\omega})$

In other words, if a time function is even, its DTFT is also even, and if a time function is odd its DTFT is also odd.

P5. Complex conjugation. $x^*[n] \Leftrightarrow X^*(e^{j\omega})$

We note that if a time function is *real*, $x[n] = x^*[n] \Leftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$.

This latter property is referred to as *Hermitian symmetry.* If $X(e^{j\omega})$ is Hermetian symmetric, its real part is even, its imaginary part is odd, its magnitude is real, and its phase angle is odd. We note further that if a real time function is even or odd, both the Hermitian symmetry and the time reversal properties constrain the transform. Specifically, if a time function is both real and even, its DTFT is also real and even. If a time function is real and odd, its DTFT is both imaginary and odd. For example, the sine wave has the transform

$$\sin(\omega_0 n) = \frac{e^{j\omega_0} - e^{-j\omega_0 n}}{2j} \Leftrightarrow \frac{\pi}{j} \sum_{r=-\infty}^{\infty} (\delta(\omega - \omega_0 - 2\pi r) - \delta(\omega + \omega_0 - 2\pi r)) \tag{23}$$

P6. Parseval's theorem.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega \tag{24}$$

This relationship is important because the quantity on the left side of the equation is clearly the total energy of the time function. Hence the energy in a frequency band can be obtained by integrating the function

$\left|X(e^{j\omega})\right|^2$ over that frequency band. (Be sure to take the negative frequencies into account!) Because of its physical meaning, the function $\left|X(e^{j\omega})\right|^2$ is sometimes referred to as the *energy density spectrum.*

P7. Initial value theorems. These trivial-to-prove theorems are sometimes confused with Parseval's theorem. They sometimes help with computation.

$$\sum_{n=-\infty}^{\infty} x[n] = X(e^{j\omega})\Big|_{\omega=0} \tag{25}$$

$$x[0] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})d\omega \tag{26}$$