1/22/24 CONVOLUTION EXAMPLES;

DT PROCESSING IN THE FREQUENCY DOMAIN: THE DTFT

(COSEP 2.6-2.7; NEXT 2.6-2.9)

CONVOLUTION SUM

$$x[n] \rightarrow \boxed{LSI} \rightarrow y[n]$$

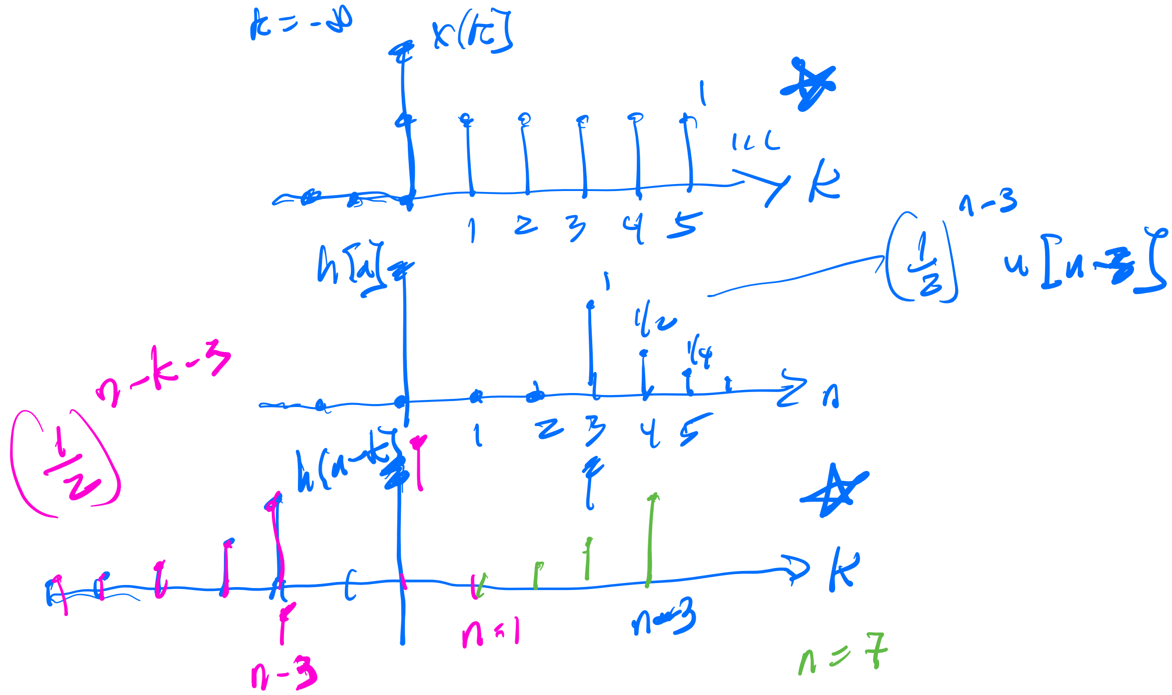
$h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] = x[n] * h[n]$$

$$x[n] = u[n]$$

$$h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$y[n] = 0, \quad n-3 < 0 \; ; \; n < 3$$

For $n \geq 3$ $\quad y[n] = \sum_{k=0}^{n-3} x[k]\, h[n-k]$

$$= \sum_{k=0}^{n-3} 1 \cdot \left(\frac{1}{2}\right)^{n-k-3} = \left(\frac{1}{2}\right)^{n-3} \cdot \sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^{-k}$$

USE $\displaystyle\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \; |\alpha| < 1 \; ; \; \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$\boxed{y[n] = 0, \; n < 3}$$

$$y[n] = \left(\frac{1}{2}\right)^{n-3} \sum_{k=0}^{n-3} 2^k = \left(\frac{1}{2}\right)^{n-3} \frac{1-2^{n-2}}{1-2}$$

$$= \left(\frac{1}{2}\right)^{n-3} \left(2^{n-2} - 1\right) = \left(\frac{1}{2}\right)^{n-3} \left(\left(\frac{1}{2}\right)^{2-n} - 1\right)$$

$$= \left(2 - \left(\frac{1}{2}\right)^{n-3}\right) u[n-3]$$

LET SYSTEM, INPUT $x[n] = e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$

$$e^{j\omega_0 n} \xrightarrow[\text{LSI}]{} \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] = \sum_{k=-\infty}^{\infty} e^{j\omega_0 (n-k)}\, h[k]$$

$$= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}\, h[k]}_{H(e^{j\omega})\big|_{\omega = \omega_0}}$$

TRANSFER FUNCTION

$$\underline{\underline{A}}\, \underline{x} = \lambda\, x$$

EIGENVALUE

EIGENFUNCTION

CONTINUOUS-TIME FOURIER TRANSFORMS (CTFTs)

BASIS FUNCTIONS

$\{e^{j\Omega t}\}$

IF $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$

ICTFT $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$

CTFT $\quad X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$

DISCRETE-TIME FOURIER TRANSFORM (DTFT)

BASIS FUNCTIONS

$\{e^{j\omega n}\}$

DTFT $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

CONSIDER $\quad e^{j\omega_0 n} = e^{j\omega n}\Big|_{\omega=\omega_0}$

IDTFT $\quad x[n] = \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} \cdot d\omega$

PERIODIC, PERIOD 2π ; PERIODIC IN ω, PERIOD 2π

NOW CONSIDER

$$e^{j\omega n}\Big|_{\omega=\omega_0+2\pi r}$$

INTEGER (r)

$$= e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}\cdot \underbrace{e^{j2\pi r n}}_{1}$$

$$= e^{j\omega_0 n}$$

$\int_{-\pi}^{\pi}$, $\int_{0}^{2\pi}$, $\int_{\langle 2\pi\rangle}$

$$\boxed{X(e^{j\omega}) \text{ IS PERIODIC IN } \omega, \text{ PERIOD } 2\pi}$$

CONSISTENCY OF IDTFT

DTFT $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

IDTFT $\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$

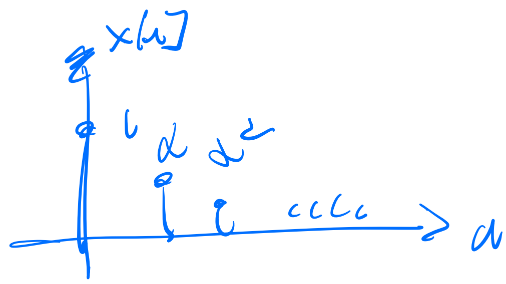
CONSIDER $\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sum_{\ell=-\infty}^{\infty} x[\ell]\, e^{-j\omega \ell}}_{X(e^{j\omega})}\, e^{j\omega n}\, d\omega$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\, \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}\, d\omega}_{= 1,\ n=\ell;\quad 0,\ n\neq \ell}$$

$$= x[n]$$

EXAMPLES of DTFTs .....

1. $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

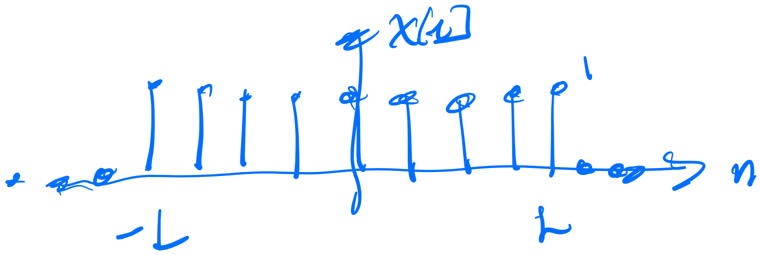


$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

2. FINITE-DURATION BOX



$$x[n] = \begin{cases} 1, & |n| \le L \\ 0, & \text{ELSE} \end{cases}$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

DTFT $\quad X(e^{j\omega}) = \sum_{n=-L}^{L} 1 \cdot e^{-j\omega n}$

$N = 2L+1$

let $l = n+L$, $n = l - L$

$$= \sum_{l=0}^{2L} 1 \cdot e^{-j\omega(l-L)}$$

$$= e^{j\omega L} \sum_{l=0}^{2L} e^{-j\omega l} = e^{j\omega L} \cdot \frac{1 - e^{-j\omega(2L+1)}}{1 - e^{-j\omega}}$$

$$X(e^{j\omega}) = e^{j\omega L} \frac{(1 - e^{-j\omega N})}{1 - e^{-j\omega}}$$

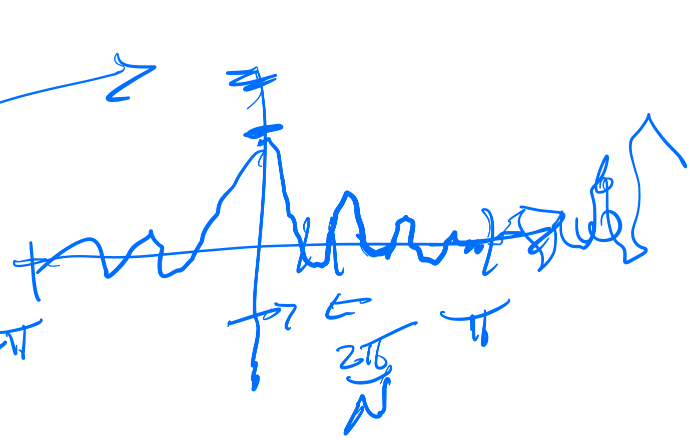
CORRECTED

$$= e^{j\omega L} \frac{e^{-j\frac{\omega N}{2}}\left(e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$L = \frac{N-1}{2}$

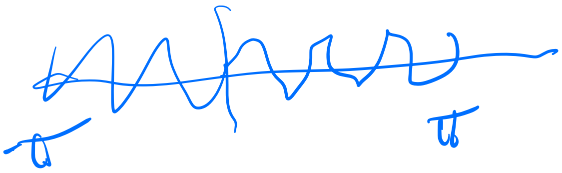
$$= \cancel{e^{j\omega\left(\frac{N-1}{2} - \frac{(N-1)}{2}\right)}} \cdot \frac{\cancel{2j}\sin\left(\frac{\omega N}{2}\right)}{\cancel{2j}\sin\left(\frac{\omega}{2}\right)} = \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

CONSIDER $\dfrac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$



$$\frac{\lim\limits_{\omega \to 0} \frac{\omega N}{2} \cos\left(\frac{\omega N}{2}\right)}{\lim\limits_{\omega \to 0} \frac{\omega}{2} \cos\left(\frac{\omega}{2}\right)} = N$$

NUM



DENOM