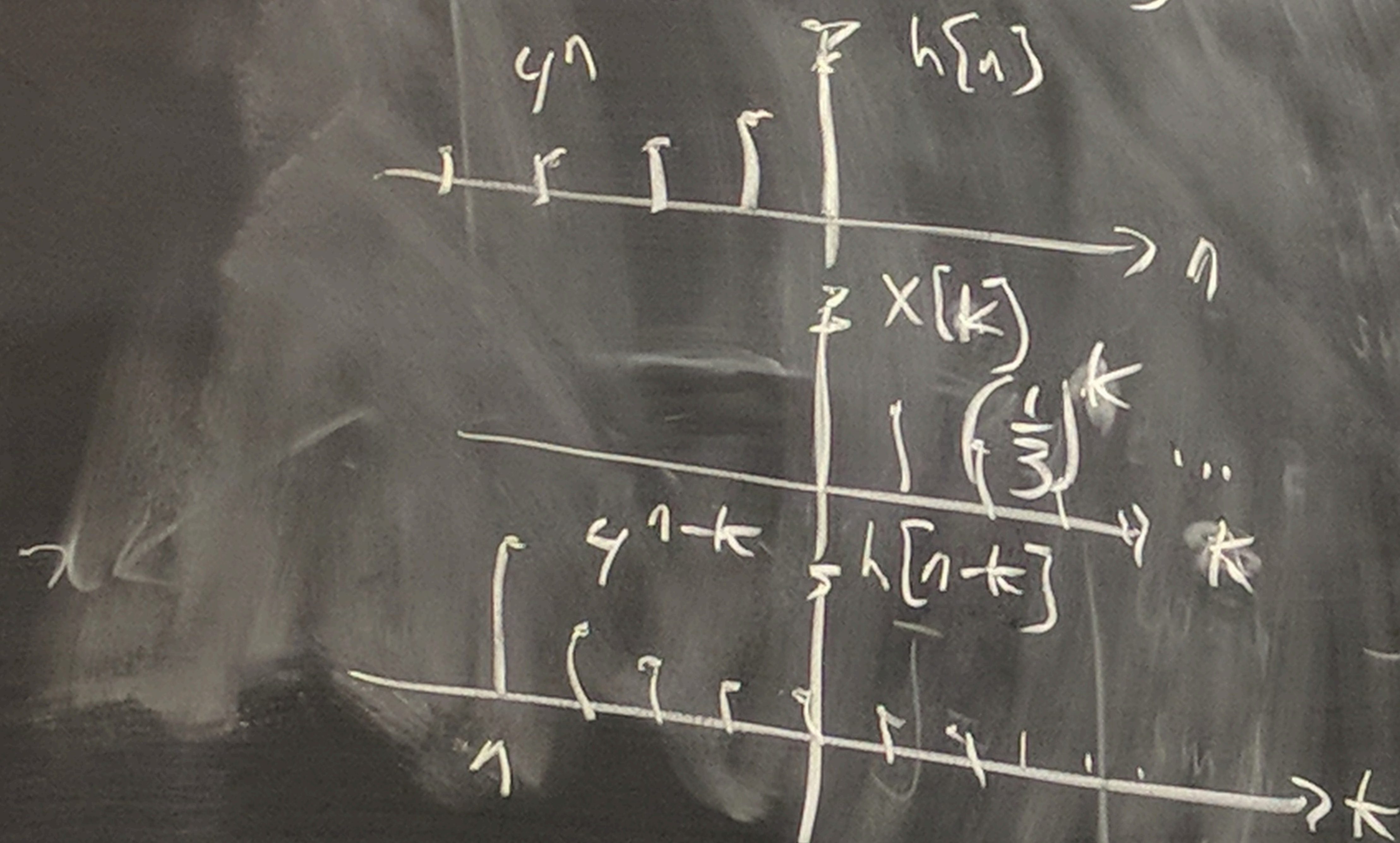


4A CONVOLUTION EXAMPLE

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$h[n] = 4^n u[-n]$$



DISCRETE-TIME PROCESSING
IN THE FREQUENCY DOMAIN:
THE DFT

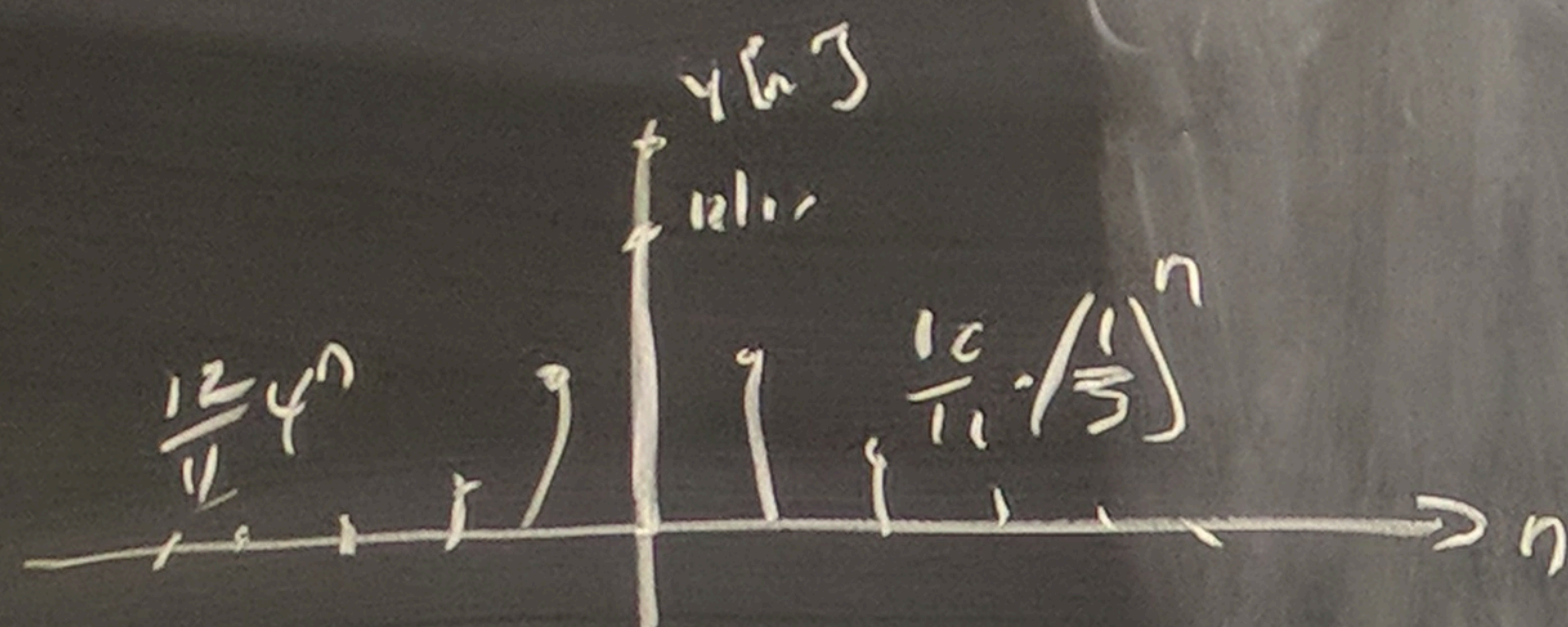
(OS4 P. 2.6-2.7)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

For $n < 0$, $y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k (4)^{n-k}$

$n \geq 0$, $y[n] = \sum_{k=-n}^{\infty} \left(\frac{1}{3}\right)^k (4)^{n-k}$

$\Delta V =$



$$\begin{aligned}
 n < 0: \\
 y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k (4)^{n-k} \\
 &= 4^n \sum_{k=0}^{\infty} \left(\frac{1}{12}\right)^k = 4^n \cdot \frac{1}{1 - \frac{1}{12}} = 4^n \cdot \frac{12}{11}
 \end{aligned}$$

$$\begin{aligned}
 n > 0 \\
 y[n] &= \sum_{k=n}^{\infty} 4^k \left(\frac{1}{12}\right)^k \\
 &= 4^n \sum_{l=0}^{\infty} \left(\frac{1}{12}\right)^{l+n} = \left(\frac{1}{3}\right)^n \cdot \sum_{l=0}^{\infty} \left(\frac{1}{12}\right)^l = \frac{12}{11} \left(\frac{1}{3}\right)^n
 \end{aligned}$$

Let $l = k - n$, $k = l + n$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad |\alpha| < 1$$

$$\sum_{n=0}^{\infty} \alpha^{2n} = \frac{1-\alpha^2}{1-\alpha}$$

CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

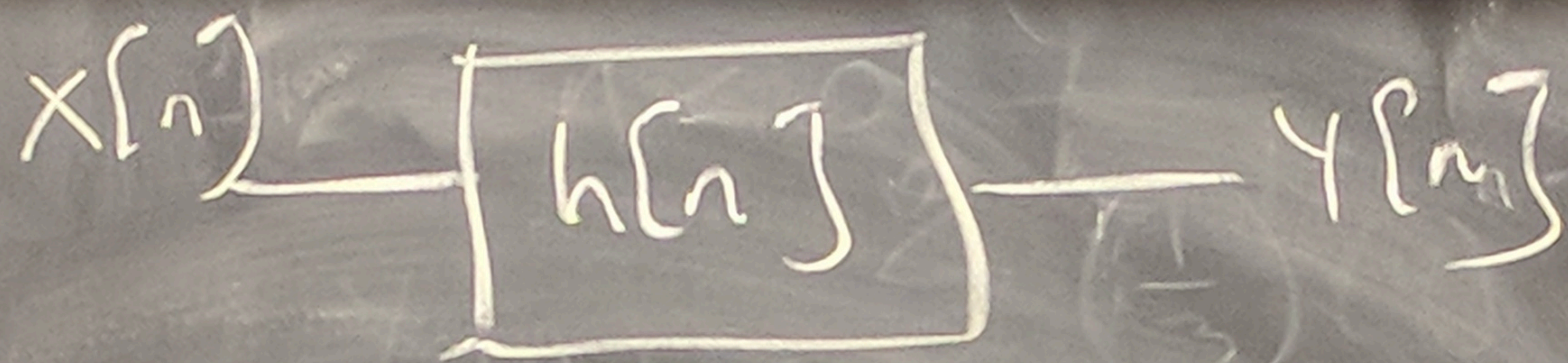
(2π)

DISCRETE-TIME PROCESSING
IN THE FREQUENCY DOMAIN:
THE DTFT
(OSYP 2.6-2.7)

$$X(e^{j\omega}) \Big|_{\omega=\omega_0} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n}$$

$$X(e^{j\omega}) \Big|_{\omega=\omega_0+2\pi} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega_0+2\pi)n}$$

$e^{-j\omega_0 n} \cdot e^{-j2\pi n}$



$$x[n] = e^{j\omega_0 n} \xRightarrow{\text{LTI}} y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

$$x[n] \rightarrow H(e^{j\omega}) \Big|_{\omega = \omega_0}$$

$$\angle X(e^{j\omega}) = \tan^{-1} \left[\frac{-\alpha \sin(\omega)}{1 - \alpha \cos(\omega)} \right] \approx \frac{-\alpha \omega}{1 - \alpha}$$

For $\omega \approx 0$, $\sin(\omega) \approx \omega$
 $\cos(\omega) \approx 1$
 $\tan(\omega) \approx \omega$, $\tan^{-1}(\omega) \approx \omega$

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1 - \alpha e^{+j\omega}}{(1 - \alpha e^{-j\omega})(1 - \alpha e^{+j\omega})}$$

$$|X(e^{j\omega})| = \left| \frac{1}{(1 - \alpha e^{-j\omega})(1 - \alpha e^{+j\omega})} \right|^{1/2}$$

$$= \sqrt{\frac{1}{1 - 2\alpha \cos(\omega) + \alpha^2}}$$

$$1 - 2\alpha \cos(\omega) + \alpha^2$$

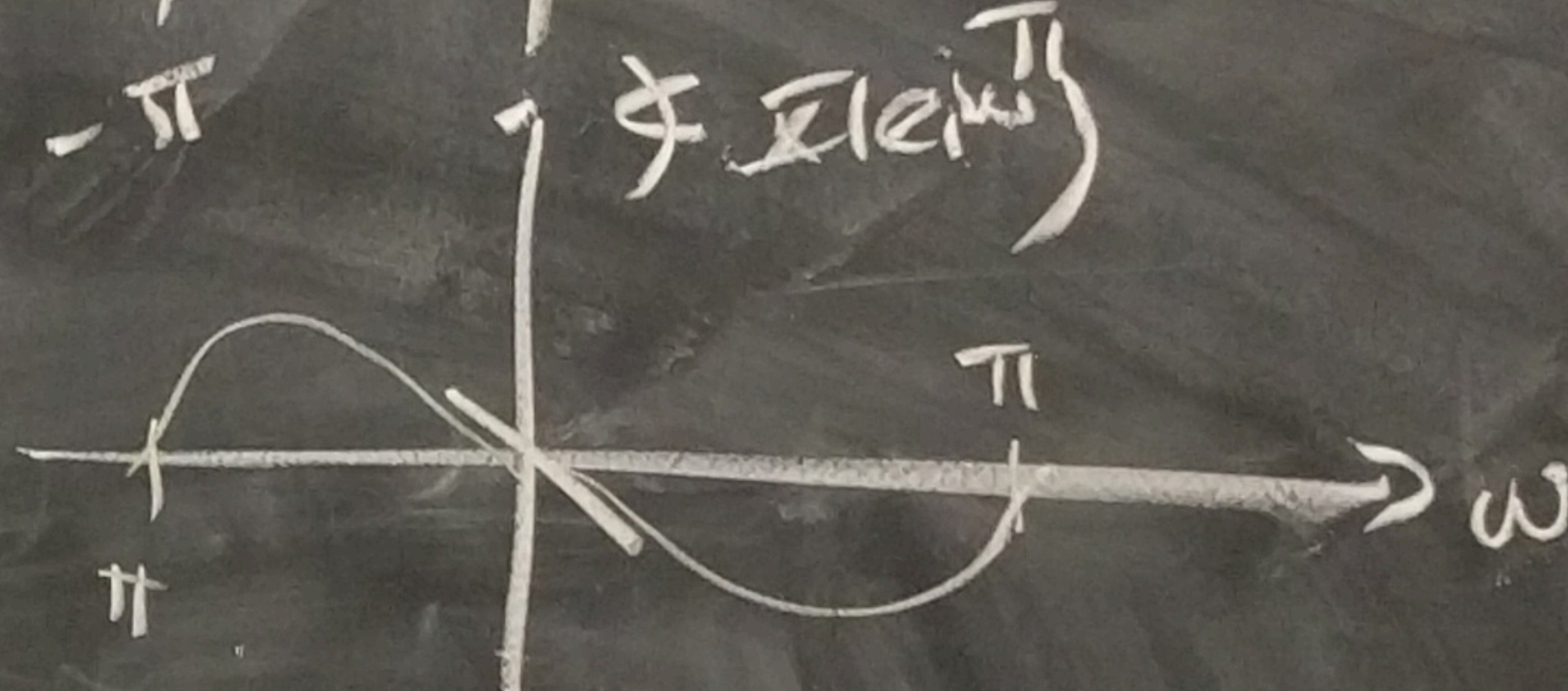
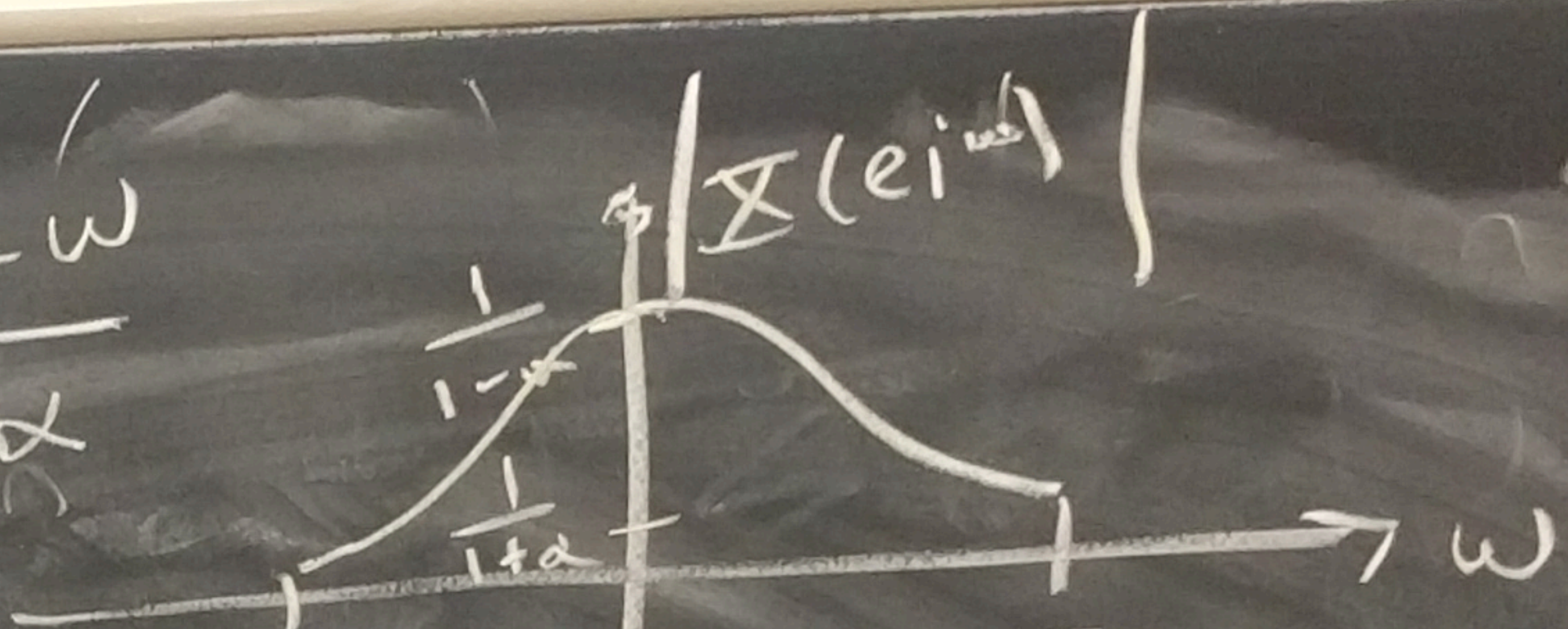
$$= \frac{1 - \alpha \cos(\omega) - j\alpha \sin(\omega)}{1 - 2\alpha \cos(\omega) + \alpha^2}$$

$$= \frac{1 - \alpha \cos(\omega) + \alpha^2}{1 - 2\alpha \cos(\omega) + \alpha^2} - j \frac{\alpha \sin(\omega)}{1 - 2\alpha \cos(\omega) + \alpha^2}$$

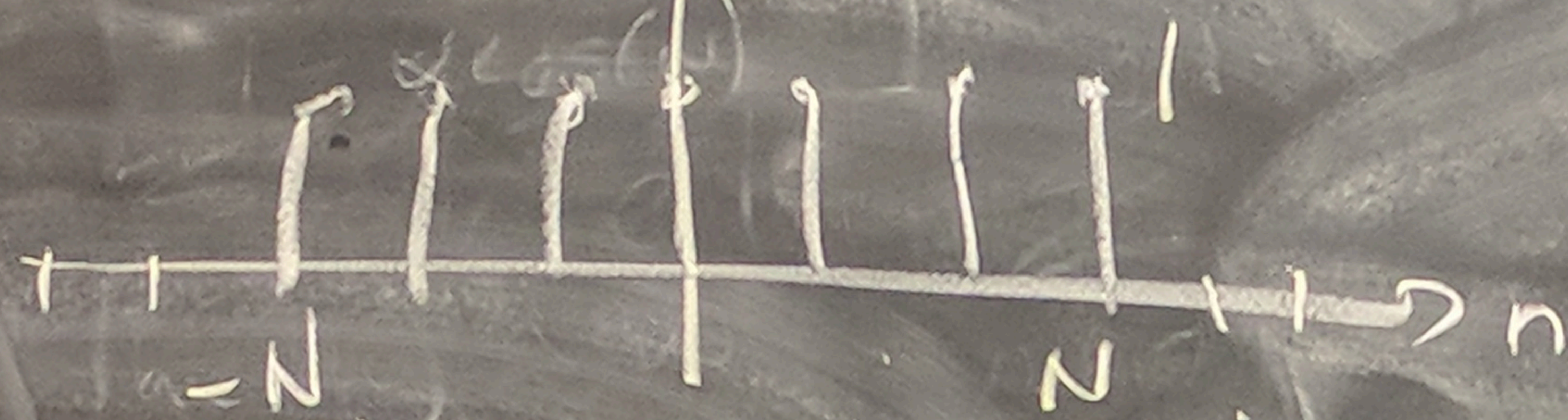
$$\angle X(e^{j\omega}) = \tan^{-1} \frac{\text{Im}[X(e^{j\omega})]}{\text{Re}[X(e^{j\omega})]}$$

$$\text{Re}[X(e^{j\omega})]$$

$$\text{Im}[X(e^{j\omega})]$$



Ex 2 $x(n)$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-N}^N 1 \cdot e^{-j\omega n}$$

$$= \sum_{l=0}^{2N} e^{-j\omega(l-N)}$$

$$l = n + N; n = l - N$$

$$= e^{j\omega N} \sum_{l=0}^{2N} e^{-j\omega l}$$

$$= e^{j\omega N} \frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}}$$

$$= \sqrt{X(e^{j\omega}) X^*(e^{j\omega})}$$

$$\begin{aligned}
 X(e^{j\omega}) &= \frac{e^{j\omega N} (1 - e^{-j\omega(2N+1)})}{1 - e^{-j\omega}} \\
 &= e^{j\omega N} \cdot \frac{1 - e^{-j\omega}}{e^{j\omega \frac{(2N+1)}{2}} (e^{+j\omega \frac{(2N+1)}{2}} - e^{-j\omega \frac{(2N+1)}{2}})} \\
 &= \cancel{e^{j\omega N}} \cancel{e^{-j\omega \frac{(2N+1)}{2}}} \frac{e^{-j\omega \frac{1}{2}} (e^{+j\omega \frac{1}{2}} - e^{-j\omega \frac{1}{2}})}{2j \sin(\frac{\omega}{2} (2N+1))} \\
 &= \cancel{e^{j\omega N}} \cancel{e^{-j\omega \frac{(2N+1)}{2}}} \frac{2j \sin(\frac{\omega}{2})}{2j \sin(\frac{\omega}{2} (2N+1))}
 \end{aligned}$$

DISCRETE-TIME PROCESSING
IN THE FREQUENCY DOMAIN:
THE DFT
(OS4P 2.6-2.7)

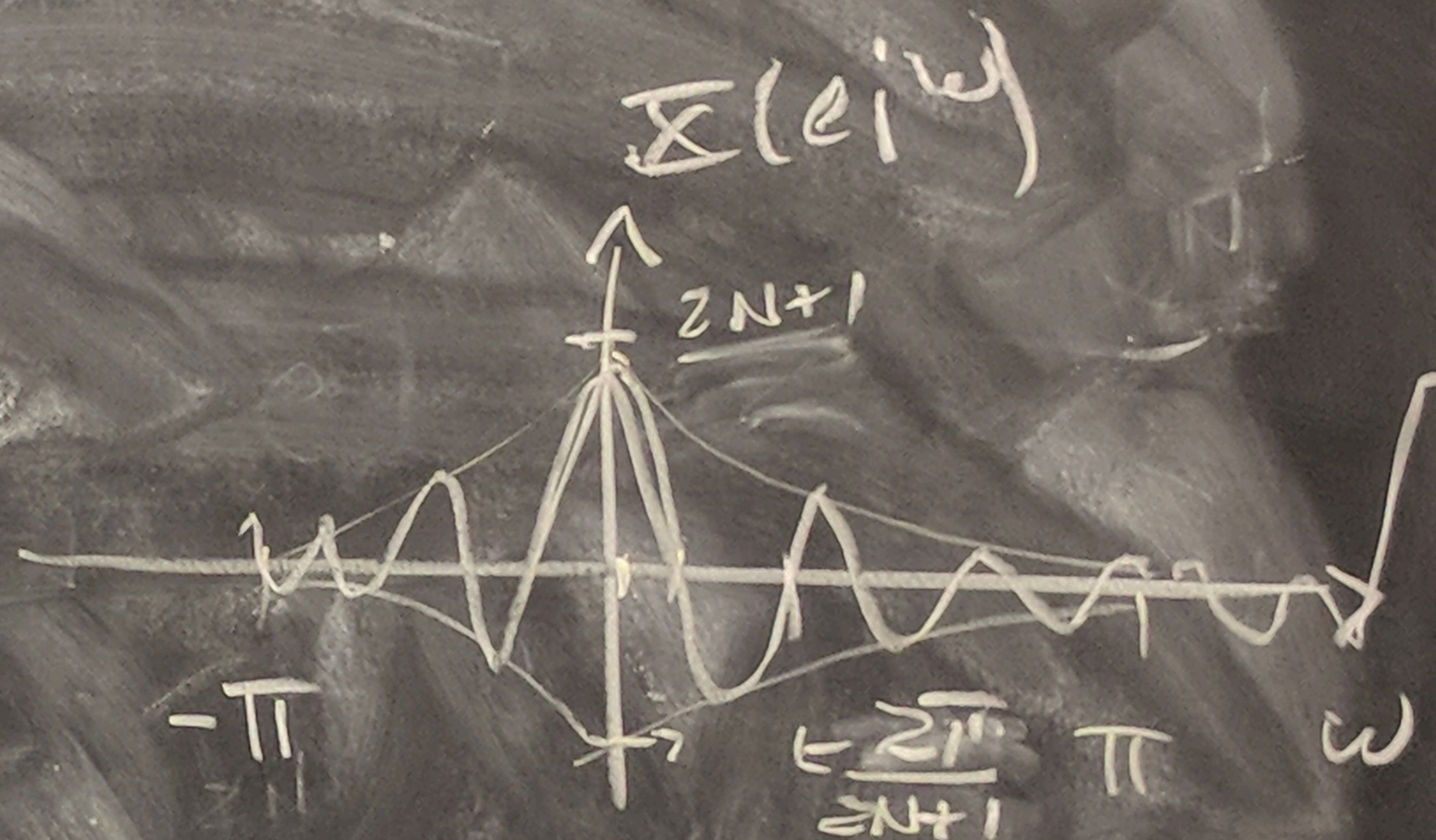
$$X(e^{j\omega}) \Big|_{\omega=\omega_0} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n}$$

$$X(e^{j\omega}) \Big|_{\omega=\omega_0+2\pi} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega_0+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n} \cdot \cancel{e^{-j2\pi n}} = X(e^{j\omega_0})$$

$\Delta V =$

$$\underline{X}(e^{j\omega}) = \frac{\sin\left(\frac{\omega}{2}(2N+1)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\sin\left(\frac{\omega}{2}\right)$$



$(2N+1)$ zeros when $\frac{\omega}{2} = k\pi$ $\omega = \frac{2N+1}{2\pi}$