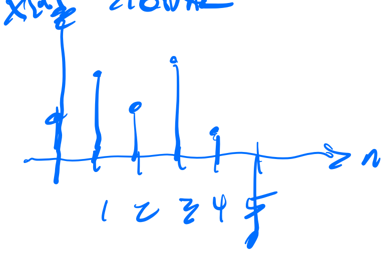
1/19/24
(B)

Review of Discrete-Time Signals + Systems
[OSYP 1, 2.0-2.4]

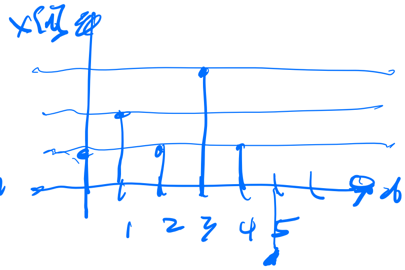
- ☆ DT Signals
- ☆ DT Systems
- ☆ Orthogonal Function Expansion
- ☆ Convolution

Continuous-Time Signals

Discrete-Time Signal



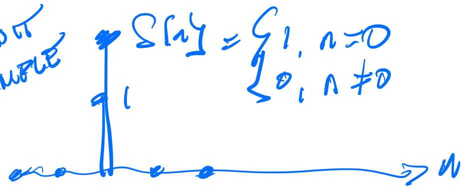
Digital Signal



Important Signals

Unit Sample

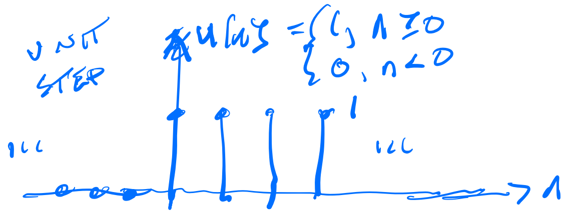
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



$$\delta[n] = u[n] - u[n-1]$$

Unit Step

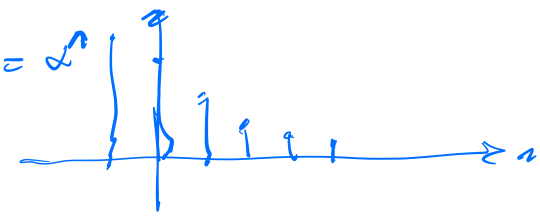
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Exponential

$$x[n] = \alpha^n$$



Periodic Time Functions

$x[n]$ is periodic with period N

if $x[n] = x[n-N]$, $\forall n$, some integer N

Ex. $x[n] = A\cos(\omega_0 n + \phi)$ $\quad \omega_0 = \frac{2\pi}{N}$

$= A\cos\left(\frac{2\pi}{N} n + \phi\right)$

$N = 5 \Rightarrow$ period 5

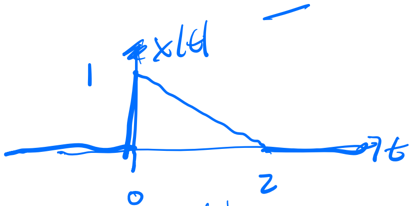
$N = \frac{3}{2} \Rightarrow$ period 3

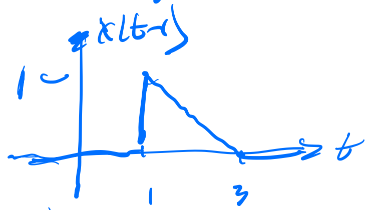
$N = \pi, \sqrt{2}, e \Rightarrow$ not periodic!

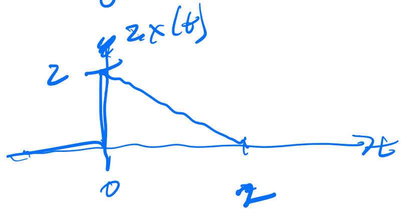
Euler $\quad e^{j\theta} = \cos(\theta) + j\sin(\theta)$

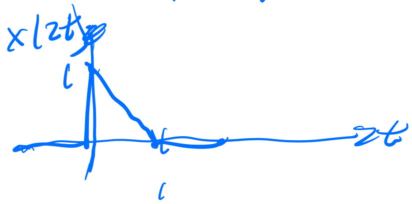
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

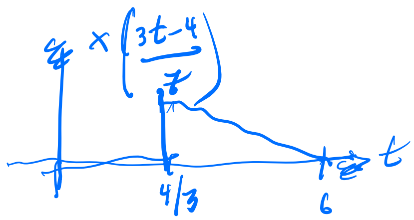
Transformation of Arguments







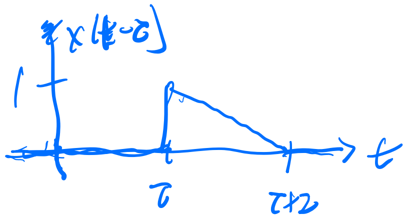


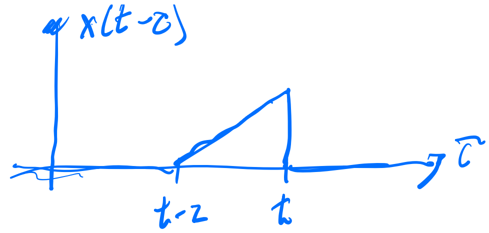


$\frac{3t-4}{7} = 0$; $3t-4=0$, $t = \frac{4}{3}$

$\frac{3t-4}{7} = 2$; $3t-4 = 14$, $3t = 18$

$t = 6$





PROPERTIES OF SYSTEMS

$$x[n] \to \boxed{T[\cdot]} \to y[n] = T\{x[n]\}$$

LINEARITY

$$T[a x_1[n] + b x_2[n]] = a T\{x_1[n]\} + b T\{x_2[n]\}$$

HOMOGENEITY

$$x[n] \to y[n]$$

THEN $a x[n] \to a\, y[n]$

SUPERPOSITION

IF $x_1[n] \to y_1[n]$

$x_2[n] \to y_2[n]$

THEN $a x_1[n] + b x_2[n] \to a y_1[n] + b y_2[n]$

TIME INVARIANCE | SHIFT INVARIANCE

TI | SI

IF $x[n] \to y[n]$

THEN $x[n-N] \to y[n-N]$, $\forall n$, $\forall$ INTEGER N

EX: L, NOT SI:

$y[n] = n \cdot x[n]$

$y[n] = x[n] \cos(\omega_0 n)$

$y[n] = x[n] \cdot w[n]$

SI, NOT L:

$y[n] = x^2[n]$

$y[n] = e^{x[n]}$

$\boxed{\text{CAUSALITY}}$

CAUSAL IF $y[n]$ DEPENDS ONLY ON VALUES OF $x[k]$, FOR $k \leq n$

$\boxed{\text{MEMORYLESSNESS}}$

MEMORYLESS IF $y[n]$ DEPENDS ONLY ON VALUES OF $x[k]$ FOR $k = n$

$\boxed{\text{STABILITY}}$

SYSTEM IS STABLE WHEN

IF $|x[n]| \leq B_1$ BIBO

THEN $|y[n]| \leq B_2$

UNSTABLE EXAMPLE

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

FOR $x[n] = u[n]$

$$y[n] = (n+1)\,u[n]$$

ORTHOGONAL FUNCTION EXPANSION

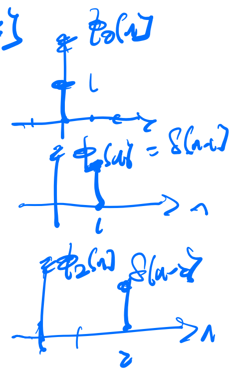
CONSIDER FUNCTIONS $\{\phi_k[n]\}$

$\{\phi_k[n]\}$ ARE ORTHONORMAL (ON) IFF

$$\sum_{n=-\infty}^{\infty} \phi_k[n]\,\phi_\ell^*[n] = \delta_{k\ell} = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

KRONECKER δ

Ex. $\phi_k[n] = \delta[n-k]$



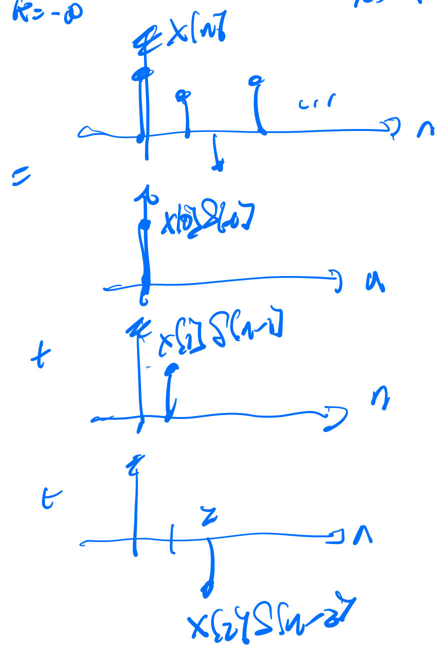
If $x[n] = \sum_{k=-\infty}^{\infty} a_k \phi_k[n]$

Ex: $\phi_k[n] = \delta[n-k]$

$$a_k = \sum_{n=-\infty}^{\infty} x[n]\, \phi_k^*[n]$$

For $\phi_k[n] = \delta[n-k]$, $\quad a_k = \sum_{n=-\infty}^{\infty} x[n]\, \delta^*[n-k] = x[k]$

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \phi_k[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

OUTPUTS of LSI SYSTEMS TO ARBITRARY INPUTS

$$x[n] \longrightarrow \boxed{\text{LSI SYSTEM}} \longrightarrow y[n]$$

DEFINE $\delta[n] \longrightarrow h[n]$ UNIT SAMPLE RESP.

$\delta[n-k] \longrightarrow h[n-k]$ SI

$x[k]\,\delta[n-k] \longrightarrow x[k]\,h[n-k]$ L, H

$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$ L, S

GIVEN LSI SYSTEM, $\delta[n] \to h[n]$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$

CONVOLUTION SUM

let $\ell = n-k$, $k = n-\ell \Rightarrow y[n] = \sum_{\ell=-\infty}^{\infty} x[n-\ell]\,h[\ell]$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] = x[n] * h[n]}$$

CONVOLUTION SUM

SOLVING CONVOLUTION ANALYTICALLY

RECALL $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$, $|\alpha| < 1$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

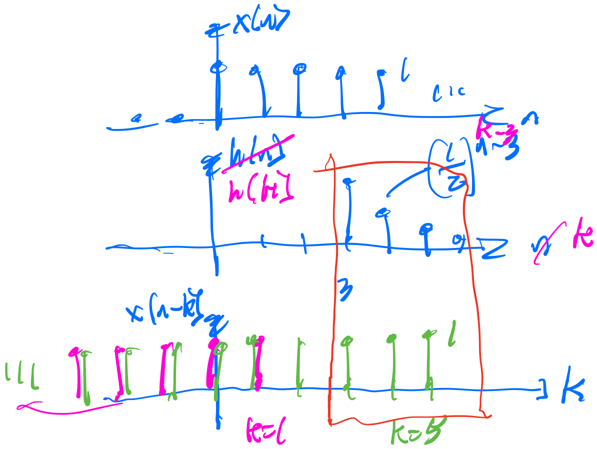
$$\sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{\infty} \alpha^n - \sum_{n=N}^{\infty} \alpha^n = \sum_{n=0}^{\infty} \alpha^n - \sum_{\ell=0}^{\infty} \alpha^{(\ell+N)}$$

let $\ell = n-N$, $n = \ell+N$, $n = N \Rightarrow \ell = 0$

$$= \sum_{n=0}^{\infty} \alpha^n - \alpha^N \sum_{\ell=0}^{\infty} \alpha^\ell = \frac{1}{1-\alpha} - \frac{\alpha^N}{1-\alpha} = \frac{1-\alpha^N}{1-\alpha}$$

Ex. $x[n] = u[n]$

$h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$



$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

$y[n] = 0, \quad n < 3$

For $n \geq 3$

$y[n] = \sum_{k=3}^{n} 1 \cdot \left(\frac{1}{2}\right)^{k-3}$

$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$y[n] = \sum_{k=3}^{n} \left(\frac{1}{2}\right)^{k-3} = \sum_{l=0}^{n-3} \left(\frac{1}{2}\right)^{l+3-3} = \frac{1-\left(\frac{1}{2}\right)^{n-2}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{n-2}\right) u[n-3]$

Let $l = k-3$, $k = l+3$

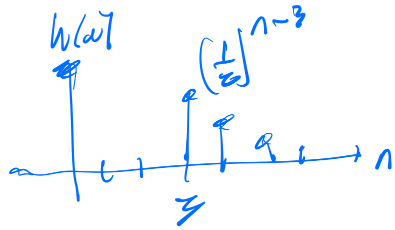
$k = 3 \Rightarrow l = 0$, $k = n \Rightarrow l = n-3$

$= \left(2 - \left(\frac{1}{2}\right)^{n-3}\right) u[n-3]$

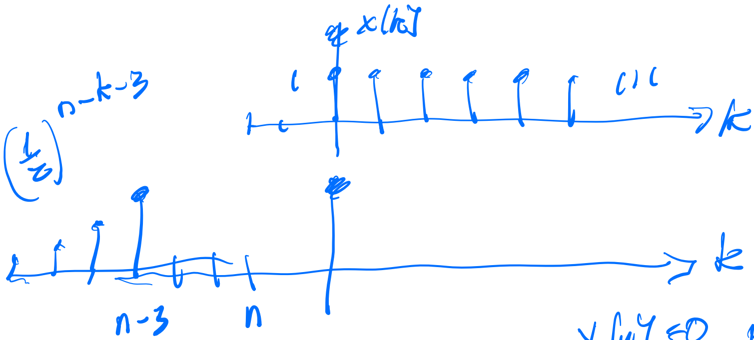
$x[n] = u[n]$, $h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$

$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$= \sum_{k=-\infty}^{\infty} u[k] \left(\frac{1}{2}\right)^{n-k-3} u[n-k-3]$



$\left(\frac{1}{2}\right)^{n-k-3}$



$y[n] = 0$ FOR $n-3 < 0$
$n < 3$

CORRECTED SOLUTION: FOR $n \geq 3$, $\displaystyle\sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^{n-k-3}$

$$= \left(\frac{1}{2}\right)^{n-3} \sum_{k=0}^{n-3} 2^k = \left(\frac{1}{2}\right)^{n-3} \frac{1-2^{n-2}}{1-2} = \left(\frac{1}{2}\right)^{n-3}\left((2)^{n-2}-1\right)$$

$$= 2 - \left(\frac{1}{2}\right)^{n-3} \Rightarrow \left(2-\left(\frac{1}{2}\right)^{n-3}\right) u[n-3]$$

CONSIDERING $\forall n$