Carnegie Mellon DSP Electrical & Computer ENGINEERING

Fundamentals of Signal Processing (18-491)
Spring Semester, 2019

COMPUTING THE CONVOLUTION SUM

Introduction

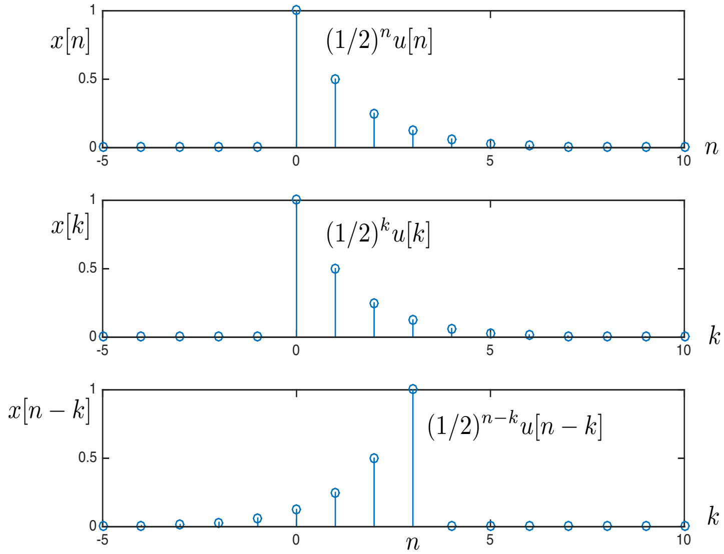
In this handout we review some of the mechanics of convolution in discrete time. This note is primarily concerned with providing examples and insight into how to solve problems involving convolution, with a few standard examples. The text provides an extended discussion of the derivation of the convolution sum and integral. These notes follow the discussion in the recitations on January 18.

Discrete-time convolution

In the lectures we showed that if an LSI system has an input $x[n]$ and a unit sample response $h[n]$ (recall that by definition $h[n]$ is whatever the system outputs when the input is $\delta[n]$), the system output will be

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

It is important to realize what the computation actually involves. In both summations, the independent variable is k, not n. The function $x[k]$ plotted as a function of k is, of course, identical to $x[n]$ plotted as a function of n except that the independent variable n is replaced by the independent variable k. The function $x[n-k]$ plotted as a function of k, on the other hand, is "flipped and shifted," or in other words time-reversed and translated along the k axis, with the part of the original function used to be at $n = 0$ now showing up along the k axis at $k = n$. The figure below compares $x[n]$, $x[k]$ and $x[n-k]$ plotted as a function of n, k, and k, respectively, for the discrete-time function $x[n] = (1/2)^n u[n]$:



(On the final line $x[n-k]$ is plotted for $n = 3$.)

Computation of the convolution sum – Example 1

Now let's consider our first real example. Specifically, we will consider the functions

$$x[n] = u[n] \text{ and}$$

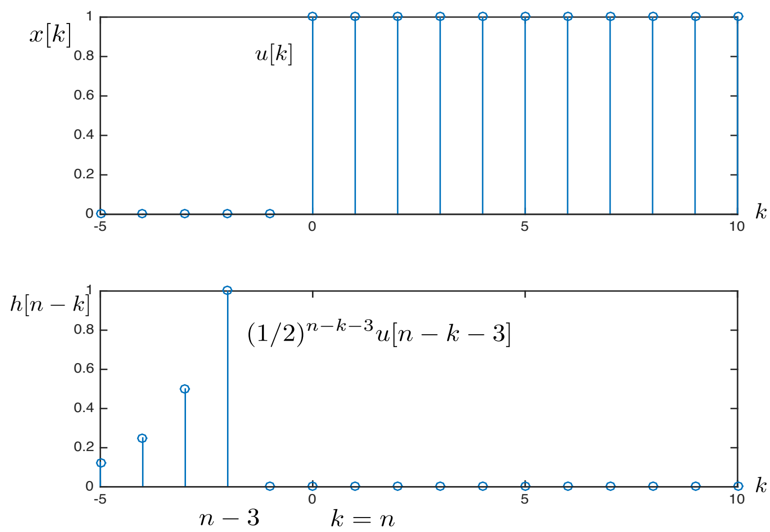
$$h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$$

This can be interpreted as an input that is a step function going into a simple lowpass filter with a delay of 3 samples. The lowpass filter will smooth the sharp edges of the input. As I mentioned in class, it is important to understand the convolution operation on many levels. We use graphical representations of the functions in the convolution sum (as demonstrated in class using MATLAB) to give us overall insight into the form of the output and the limits of non-zero output points. We will first consider the expression from the graphical standpoint and then solve analytically.

Graphical analysis. To obtain the result of the convolution we must first decide which function to flip and shift. Normally we are more likely to flip and shift the simpler of the two functions being convolved. Here, though, for purposes of illustration, we will flip and shift the more complicated function $h[n]$, causing the convolution sum to be written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The functions $x[k]$ and $h[n-k]$ are illustrated below for $n = 1$.



Note that $h[n-k]$ is developed by replacing the n in $h[n]$ by $n-k$ wherever it appears. Because the original function, $h[n] = \left(\frac{1}{2}\right)^{n-3}u[n-3]$, begins at $n = 3$, the function $h[n-k]$ *ends* three samples to the left of the sample $k = n$. Note also that the two functions $x[k]$ and $h[n-k]$ will have no nonzero overlap at all unless $n-3$ is greater than zero, or $n \geq 3$. For $n \geq 3$ the sum on k will be evaluated for $0 \leq k \leq n-3$.

Analytical computation. Computation of convolution sums involving exponentials (including step functions and sine/cosine functions) frequently involve the use of the geometric series in its infinite or finite form:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ which converges only for } |\alpha| < 1, \text{ and}$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ which is a finite sum and hence always converges}$$

For the convolution sum we are considering right now we have for $n \geq 3$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^{n-k-3}$$

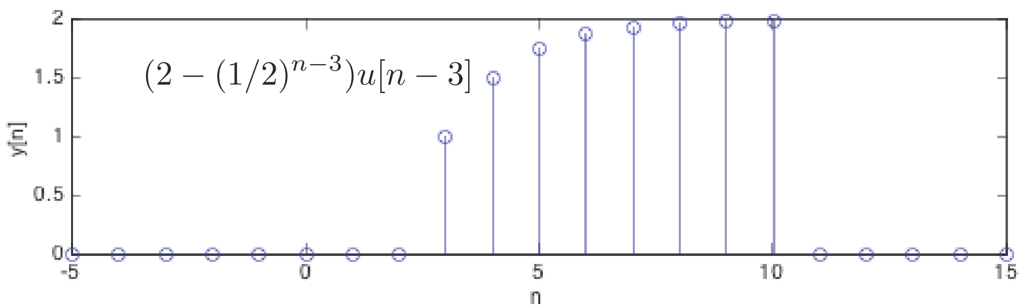
$$y[n] = \left(\frac{1}{2}\right)^{n-3} \sum_{k=0}^{n-3} 2^k = \left(\frac{1}{2}\right)^{n-3} \frac{(1-2^{n-2})}{(1-2)}$$

$$y[n] = \left(\frac{1}{2}\right)^{n-3}\left(\left(\frac{1}{2}\right)^{2-n} - 1\right) = 2 - \left(\frac{1}{2}\right)^{n-3}$$

Hence, for all values of n we can write

$$y[n] = \left(2 - \left(\frac{1}{2}\right)^{n-3}\right)u[n-3]$$

This function is illustrated below:

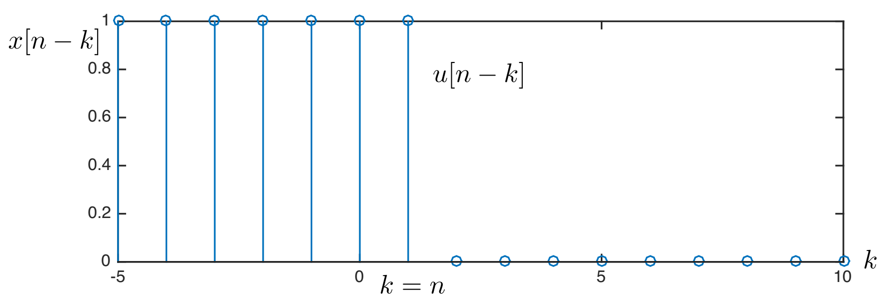


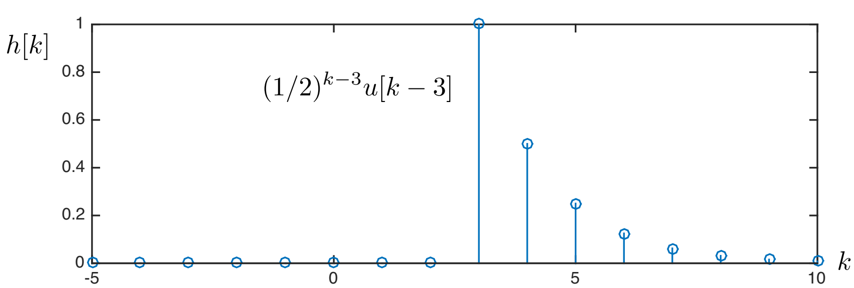
Alternate solution. Now let's suppose that we flip and shift $x[n]$ instead of $h[n]$, which is what might normally be done because the input function is simpler in form:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] \text{ where}$$

$$x[n-k] = u[n-k]$$

These functions are depicted below, again for $n = 1$.





Because the function $h[k]$ does not begin until $k = 3$, the product of $x[n-k]$ and $h[k]$ will be nonzero only for $n \geq 3$, as before. For values of $n \geq 3$, the output is computed as

$$y[n] = \sum_{k=3}^{n} \left(\frac{1}{2}\right)^{k-3}$$

Because our expressions for geometric series are defined only for an initial term at the index $k = 0$, we perform a change of variable. Let $l = k-3$; $k = l+3$. It is easy to see that when $k = 3$, $l = 0$ and when $k = n$, $l = n-3$. Hence, we can rewrite the sum as

$$y[n] = \sum_{k=3}^{n} \left(\frac{1}{2}\right)^{k-3} = \sum_{l=0}^{n-3} \left(\frac{1}{2}\right)^{l+3-3} = \sum_{l=0}^{n-3} \left(\frac{1}{2}\right)^{l+3-3}$$

This evaluates to

$$y[n] = \sum_{l=0}^{n-3} \left(\frac{1}{2}\right)^{l} = \frac{1-\left(\frac{1}{2}\right)^{n-2}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{n-2}\right) = \left(2-\left(\frac{1}{2}\right)^{n-3}\right) \text{ for } n \geq 3$$

or $$y[n] = \left(2-\left(\frac{1}{2}\right)^{n-3}\right)u[n-3]\text{, as before, of course.}$$

These calculations were illustrated in the class demos from January 26, which are available on Piazza. Specifically, the calling scripts The calling scripts `run4.m` and `run5 .m` flip and shift $x[n]$ and $h[n]$, respectively (although the notation on the screens may be a little different).

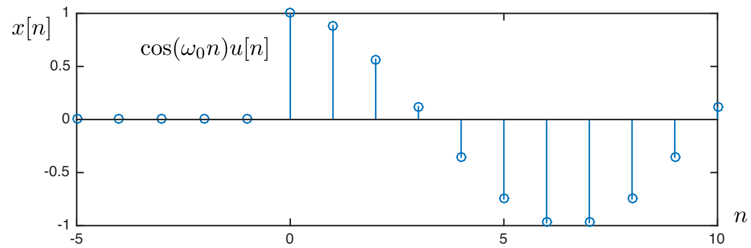
Computation of the convolution sum – Example 2

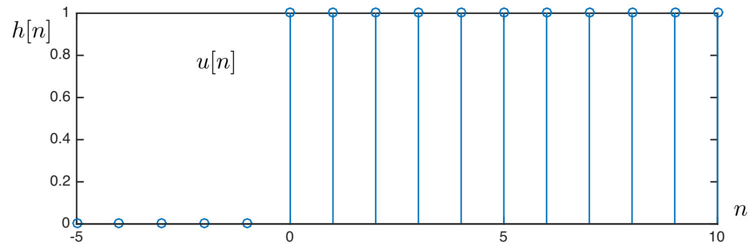
Let's consider the convolution of an arbitrary $x[n]$ with a sample response that is simply the delta function $\delta[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The sum above will be zero for all values of k except for $k = n$, so the result of the sum will be $y[n] = x[n]$. In other words, $x[n] * \delta[n] = x[n]$, or (in words), delta functions are the identity function under the operation of convolution. Similarly, convolving an arbitrary function with a shifted delta function will produce the arbitrary function shifted by the same amount as the delta function is shifted.

Computation of the convolution sum – Example 3





Now consider the convolution of $x[n] = \cos(\omega_0 n)u[n]$ with $h[n] = u[n]$.

To solve this we will make us of three techniques that were discussed in the recitation:

- **Technique 1.** Representing sines and cosines in complex exponential form:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- **Technique 2.** Infinite and finite summation of exponentials:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ for } |\alpha| < 1$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ for all } \alpha$$

- **Technque 3.** Balancing equations with complex exponentials:

$$(1-e^{j\omega n}) = e^{j\omega n/2}(e^{-j\omega n/2} - e^{j\omega n/2}) = -2je^{j\omega n/2}\sin(\omega/2)$$

(OK, we actually did not do this method in class before, but it is straightforward.)

As discussed in the recitation, $y[n] = 0$ for $n < 0$ because there is no value of k in the convolution sum for which both $x[k]$ and $h[n-k]$ are nonzero. For $n \geq 0$ we obtain using the convolution sum

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} \cos(\omega_0 k)$$

Applying Technique 1, we obtain

$$y[n] = \sum_{k=0}^{n} \cos(\omega_0 k) = \frac{1}{2}\sum_{k=0}^{n} e^{j\omega_0 k} + \frac{1}{2}\sum_{k=0}^{n} e^{-j\omega_0 k}$$

Applying Technique 2 (not worrying about convergence because the sum is finite), we obtain

$$y[n] = \frac{1}{2}\sum_{k=0}^{n} e^{j\omega_0 k} + \frac{1}{2}\sum_{k=0}^{n} e^{-j\omega_0 k} = \frac{1}{2}\frac{\left(1-e^{j\omega_0(n+1)}\right)}{\left(1-e^{j\omega_0}\right)} + \frac{1}{2}\frac{\left(1-e^{-j\omega_0(n+1)}\right)}{\left(1-e^{-j\omega_0}\right)}$$

Applying Technique 3, we obtain

$$y[n] = \frac{1}{2}\frac{e^{j\omega_0\left(\frac{n+1}{2}\right)}\left(e^{-j\omega_0\left(\frac{n+1}{2}\right)} - e^{j\omega_0\left(\frac{n+1}{2}\right)}\right)}{e^{j\omega_0/2}\left(e^{-j\omega_0/2} - e^{j\omega_0/2}\right)} + \frac{1}{2}\frac{e^{-j\omega_0\left(\frac{n+1}{2}\right)}\left(e^{j\omega_0\left(\frac{n+1}{2}\right)} - e^{-j\omega_0\left(\frac{n+1}{2}\right)}\right)}{e^{-j\omega_0/2}\left(e^{j\omega_0/2} - e^{-j\omega_0/2}\right)}$$

$$= \frac{1}{2}\frac{e^{j\omega_0 n/2} - 2j\sin\left(\frac{\omega_0(n+1)}{2}\right)}{-2j\sin\left(\frac{\omega_0}{2}\right)} + \frac{1}{2}\frac{e^{-j\omega_0 n/2}2j\sin\left(\frac{\omega_0(n+1)}{2}\right)}{2j\sin\left(\frac{\omega_0}{2}\right)}$$

Combining terms, we obtain

$$y[n] = \frac{\sin\left(\frac{\omega_0(n+1)}{2}\right)}{\sin\left(\frac{\omega_0}{2}\right)}\cos\left(\frac{\omega_0 n}{2}\right) \text{ for } 0 \le n$$

Note that this result *must* be real, as the two functions in the convolution sum are real.