Examples of the Convolution Computation

Introduction

These notes briefly review the convolution examples presented in class.

Computation of the convolution sum   Example 1

As I mentioned in class, it is important to understand the convolution operation on many levels. We use graphical representations of the functions in the convolution sum (as demonstrated in class using MATLAB) to give us overall insight into the form of the output and the limits of non-zero output points. In this section we will provide an example of how the convolution sum is computed analytically.

Computation of actual convolution sums in 18-491/691 most commonly involves generalized exponential functions (including step functions and the sine/cosine functions). Two mathematical relationships that are frequently used are

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ which converges only for } |\alpha| < 1, \text{ and}$$

$$\sum_{n-0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ which is a finite sum and hence always converges}$$

Now lets consider the following pair of functions to be convolved:
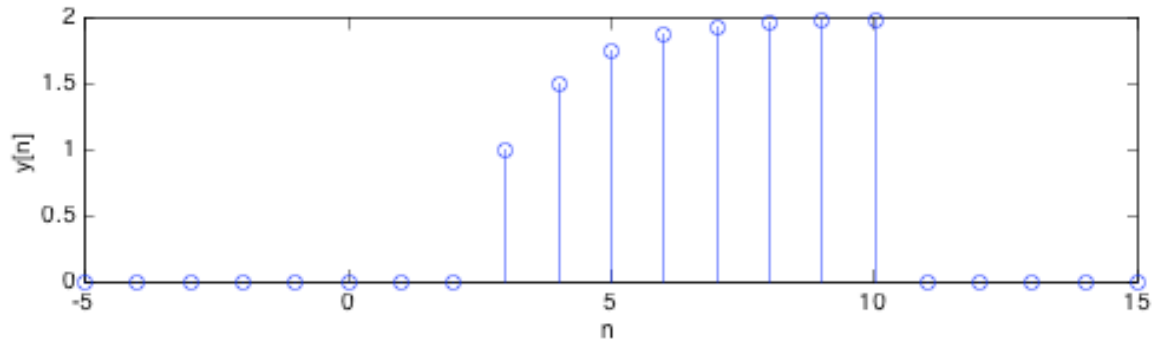
$$x[n] = i[n], \text{ and}$$

$$h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$$

The graphical convolution of these function is illustrated in the MATLAB routines that are available in

 http://www.ece.cmu.edu/ ee491/lectures

The calling scripts go4.m and go5.m flip and shift $x[n]$ and $h[n]$, respectively (although the notation on the screens may be a little different).

Note that either way, the result of the convolution looks like

To obtain this result analytically we must first decide which function to flip and shift. Normally wed flip and shift the "simpler time function, which in this case would be $x[n]$, causing the convolution sum to be written as

$$y[n] = \sum_{n=-\infty}^{\infty} h[k]x[n-k]$$

Strictly for the purpose of being more illustrative, though, we will flip and shift the more complicated function $h[n]$, causing the convolution sum to be written as

$$y[n] = \sum_{n=-\infty}^{\infty} x[k]h[n-k]$$

Given our definitions above for $x[n]$ and $h[n]T$, we can write by substitution

$$h[n-k] = \left(\frac{1}{2}\right)^{n-k-3} u[n-k-3]$$

$$x[k] = u[k]$$

Since $x[k]$ equals zero for x< 0 and $h[n-kl]$ equals zero for $k > n-3$, we can see that the only region of "overlap" for which both $x[k]$ and $h[n-k]$ are nonzero is $0 \le k \le n-3$. Hence the convolution sum is equal to zero for values of n that are less than 3. For $3 \le n$ we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n-3} 1\left(\frac{1}{2}\right)^{n-k-3}$$

$$y[n] = \left(\frac{1}{2}\right)^{n-3} \sum_{k=0}^{n-3} 2^k = \left(\frac{1}{2}\right)^{n-3} \frac{(1-2^{n-2})}{(1-2)} = \left(\frac{1}{2}\right)^{n-2} \left(\left(\frac{1}{2}\right)^{2-n} - 1\right) = 2 - \left(\frac{1}{2}\right)^{n-3}$$

So, for all n we can write

$$y[n] = \left(2 - \left(\frac{1}{2}\right)^{n-3}\right) u[n-3]$$

Computation of the convolution sum   Example 2
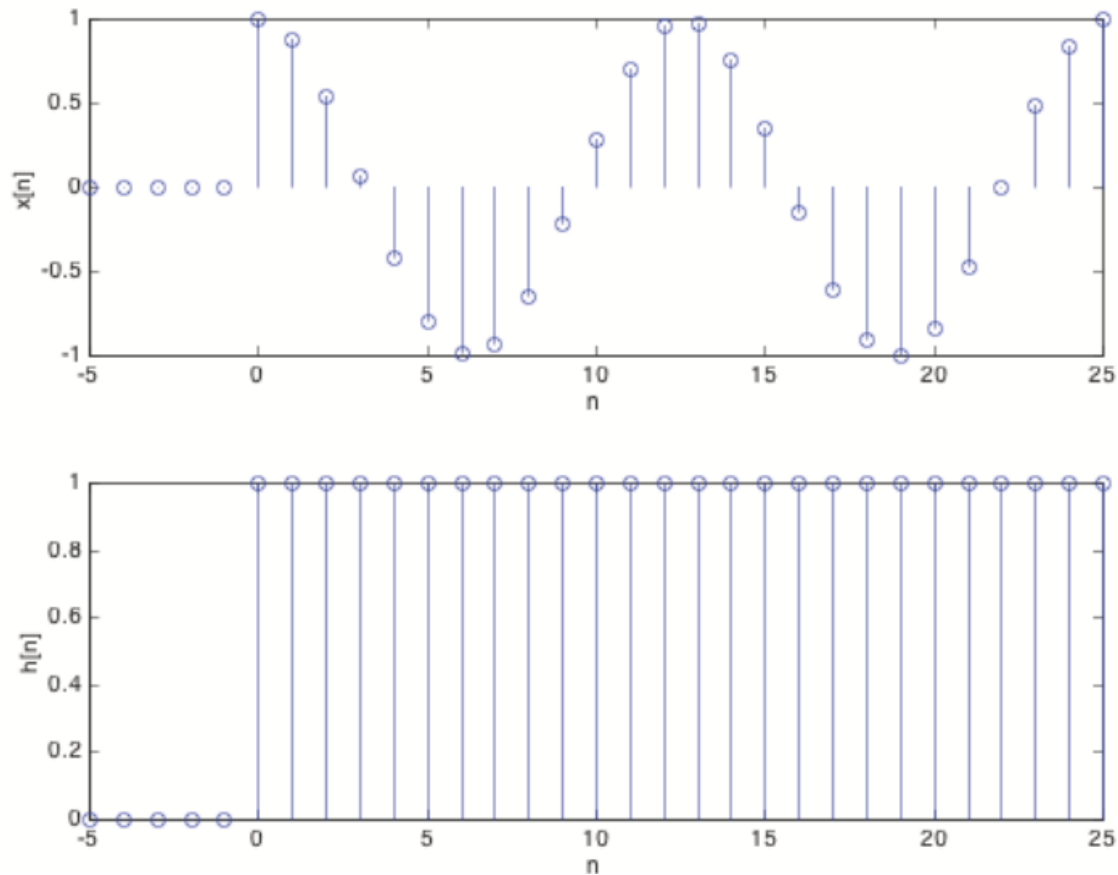
Now consider the convolution of $x[n] = \cos(\omega_0 n)u[n]$ with $h[n] = u[n]$.

First, lets recall the three "tricks" to be invoked that we reviewed in the lecture:

- **Trick 1.** Representing sines and cosines in complex exponential form:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- **Trick 2.** Infinite and finite summation of exponentials: As noted above,

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ which converges only for } |\alpha| < 1, \text{ and}$$

$$\sum_{n-0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ which is a finite sum and hence always converges}$$

- **Trick 3.** Balancing equations with complex exponentials:

$$(1 - e^{j\omega n}) = e^{j\omega n/2}(e^{-j\omega n/2} - e^{j\omega m/2}) = -2j e^{j\omega n/2}\sin(\omega/2)$$

As noted in class, $yy[n] = 0$ for $n < 0$. For $n \geq 0$ we obtain using the convolution sum

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} \cos(\omega_0 k)$$

k=0 k=0 Applying Trick 1, we obtain

$$y[n] = \sum_{k=0}^{n} \cos(\omega_0 k) = \frac{1}{2}\sum_{k=0}^{n} e^{j\omega_0 k} + \frac{1}{2}\sum_{k=0}^{n} e^{-j\omega_0 k}$$

Applying Trick 2 (not worrying about convergence because the sum is finite), we obtain

$$y][n] = \frac{1}{2}\sum_{k=0}^{n} e^{j\omega_0 k} + \frac{1}{2}\sum_{k=0}^{n} e^{-j\omega_0 k} = \frac{1}{2}\frac{\left(1 - e^{j\omega_0(n+1)}\right)}{(1 - e^{j\omega_0})} + \frac{1}{2}\frac{\left(1 - e^{-j\omega_0(n+1)}\right)}{(1 - e^{-j\omega_0})}$$

Applying Trick 3, we obtain

$$y[n] = \frac{1}{2}\frac{e^{j\omega_0(n+1)/2}\left(e^{-j\omega_0(n+1)/2} - e^{j\omega_0(n+1)/2}\right)}{\left(e^{j\omega_0 n}\left(e^{-j\omega_0/2} - e^{j\omega_0/2}\right)\right)} + \frac{1}{2}\frac{e^{-j\omega_0(n+1)/2}\left(e^{j\omega_0(n+1)/2} - e^{-j\omega_0(n+1)/2}\right)}{\left(e^{-j\omega_0 n}\left(e^{j\omega_0/2} - e^{-j\omega_0/2}\right)\right)}$$

$$y[n] = \frac{1}{2}\frac{e^{j\omega_0 n/2}(-2j\sin((\omega_0(n+1)/2))}{-2j\sin(\omega_0/2)} + \frac{1}{2}\frac{e^{-j\omega_0 n/2}(2j\sin((\omega_0(n+1)/2))}{2j\sin(\omega_0/2)}$$

Combining terms, we obtain for all n

$$y[n] = \frac{\sin(\omega_0(n+1)/2)\cos(\omega_0 n/2)}{\sin(\omega_0/2)}u[n]$$

Note that this result must be real, as the two functions in the convolution sum are real, despite all the complex exponentials flying around along the way.