Carnegie Mellon **DSP** Electrical & Computer ENGINEERING

Digital Signal Processing (18-491/18-691)
Spring Semester, 2025

QUIZ 3

Three hours | **Closed book**
April 29, 2025 | **Three pages of notes**

We list below the ground rules governing the quiz:

- The exam is closed book. You may bring three pages of notes, written on both sides, and transform tables and transform properties will be provided as they were in Quiz 2.
- You may use a traditional scientific calculator (*i.e.* with arithmetic functions, trig functions, logs, and exponents) or the calculator on your telephone if you wish. Please keep your phones on airplane mode. Texting and phone calls during the exam are strictly prohibited.

Reminder: Always try to think about ways of answering the questions using reasoning, transform properties, etc., before resorting to brute-force solutions. Please note that the questions are ordered more or less in the sequence that the topics appeared in the course, and they may not be in order of difficulty. In addition, to the greatest extent possible subparts of the questions will be graded independently based on your prior answers to the question. You are strongly encouraged to read through the whole exam before beginning your work.

Question 1: (20%):

(a) A noncausal LSI system has the unit sample response

$$h[n] = \left(\frac{1}{2}\right)^{|n|} \text{ for all } n$$

The system has the discrete-time input

$$x[n] = \left(\frac{3}{4}\right)^{n-5} u[n-5]$$

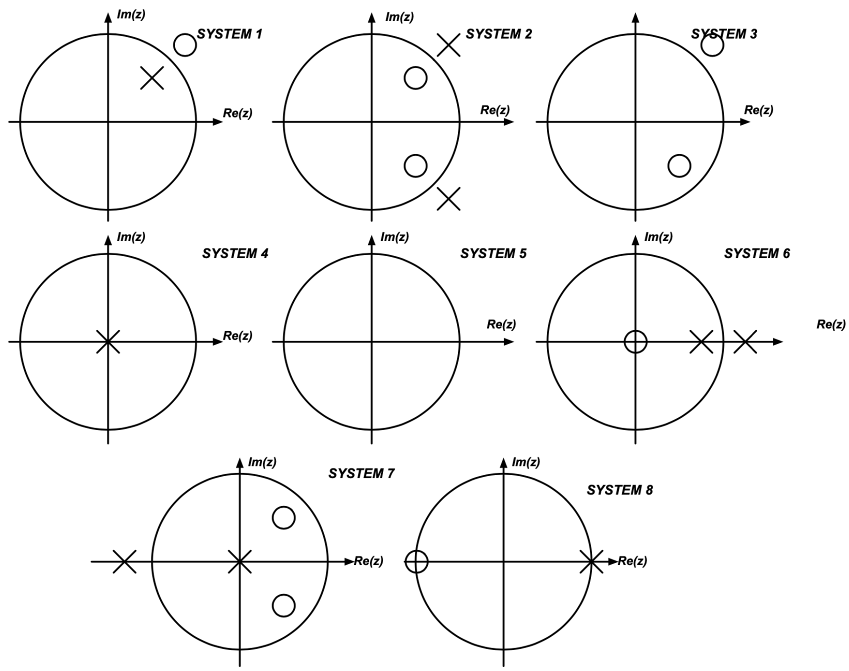
Obtain an analytical closed-form solution (no unevaluated sums or integrals) for the system output $y[n]$.

Big hint: The fastest way to the solution is to make use of z-transform techniques, calculating the inverse z-transforms using the residue method of partial fraction evaluation. Be sure to make use of the obvious properties. You do not have to solve the problem this way, but other approaches will be much more time consuming.

(b) Write the difference equation of a second system $H_2(z)$ that when cascaded with $H(z)$ produces a new system that is *allpass*. $H_2(z)$ has a maximum of two poles or zeros, and they may be anywhere in the z-plane. Please do not propose the trivial solution of the inverse system $H_I(z) = 1/H(z)$.

(c) Write the difference equation of a third system $H_3(z)$ that when cascaded with $H(z)$ produces a new system that is *minimum phase*. $H_3(z)$ has a maximum of two poles or zeros, and they may be anywhere in the z-plane.. We would like to maintain the same magnitude of the DTFT so that $|H(e^{j\omega})| = |H(e^{j\omega})H_3(e^{j\omega})|$.

Question 2: (15%):



The figure above depicts the pole-zero plots of eight different discrete-time LSI systems. While the axes are not labelled, the magnitudes of the pole and zero locations are all 0, 3/4, 1, and 4/3, while the phase angles of the pole and zero locations are all 0, $\pm\pi/4$, and π. Please answer the following questions about Systems 1 through 8 by filling in the entries of the Table on the last page with a "Y" or "N," depending on whether or not the properties considered are exhibited by the system in question. Assume that the ROC of each system includes $|z| = 0.9$.

(a) Which systems have a unit sample response $h[n]$ that is real?

(b) Which systems are causal?

(c) Which systems are stable?

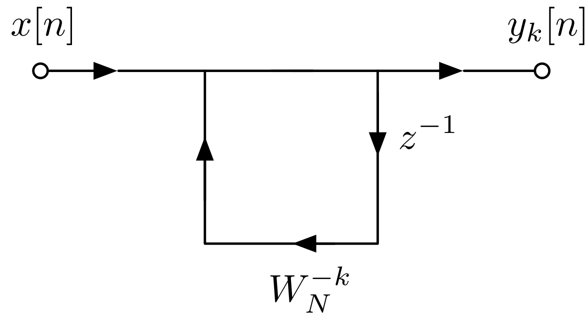
(d) Which systems are memoryless?

(e) Which systems are allpass?

(f) Which systems are linear phase?

(g) Which systems are minimum phase? (In solving this part of the question, please assume that minimum-phase system may include poles and zeros on the unit circle.)

Question 3: (15%):



The LSI discrete-time system depicted above is at the heart of the *Goertzel algorithm*, which was an early method proposed to perform the DFT operation efficiently as a convolution operation.

(a) Write the difference equation that relates the output $y_k[n]$ to the input $x[n]$. State the initial conditions for the system if $x[n] = 0$ for $n < 0$.

(b) Write an expression for the transfer function of the system,

$$H_k(z) = \frac{Y_k(z)}{X(z)}$$

(c) Using any method, obtain an expression for $h_k[n]$, the unit sample response of the system, assuming that the system is causal.

(d) If the system is causal, is it also stable? If not, how could you modify the difference equation slightly to make the system stable?

(e) Now assume that the input signal $x[n]$ is nonzero only for $0 \leq n \leq N-1$. Using the expression for the convolution sum

$$y_k[n] = \sum_l x[l] h_k[n-l]$$

Obtain an expression in sum form for the output signal $y_k[N] = y_k[n]|_{n=N}$

Compare the result you obtained with the k^{th} coefficient of the N-point DFT of $x[n]$, $X[k]$.

(f) If $x[n]$ is a finite duration sequence of length N, how many multiplication operations are needed to calculate all N coefficients for the N-point DFT of $x[n]$?

Question 4: (10%) (This is a question about DFT properties.)

Let $X(e^{j\omega})$ denote the DTFT of the sequence $x[n]$ where

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

Let $y[n]$ denote the finite-length sequence of length 10 (over the range $0 \leq n \leq 9$). $y[n]$ is obtained by computing the inverse 10-point DFT of the 10 equally-spaced samples of $X(e^{j\omega})$.

In other words,

$$y[n] = \frac{1}{10}\sum_{k=0}^{9} Y[k] W_{10}^{-nk}$$

where

$$Y[k] = X(e^{j\omega})\big|_{\omega=\frac{2\pi k}{10}}$$

(a) Obtain a set of numerical values for $y[n]$. [This is not hard if you consider the actual effect of the sampling of $X(e^{j\omega})$.]

(b) How does your answer to (a) change if we multiply the DFT coefficients by $e^{-j6\pi k/10}$ before calculating the inverse DFT? In other words, how does $y'[n]$ compare to $y[n]$ if

$$y'[n] = \frac{1}{10}\sum_{k=0}^{9} e^{-j6\pi k/10} Y[k] W_{10}^{-nk}$$

[You can answer this question even if you were unable to obtain a numerical answer for part (a).]

Question 5: (20%):

In this and the following problem we will design a bandpass filter, first using IIR techniques and subsequently using FIR techniques.

We will be developing the IIR bandpass filter by designing an IIR lowpass filter and multiplying it by a cosine to create an IIR bandpass filter using the relationship

$$h_{BP}[n] = h_{LP}[n] 2\cos(\omega_c n)$$

Specifically, we will design the IIR lowpass filter and then multiply the lowpass filter unit sample response by the cosine to get the unit sample response for the bandpass filter.

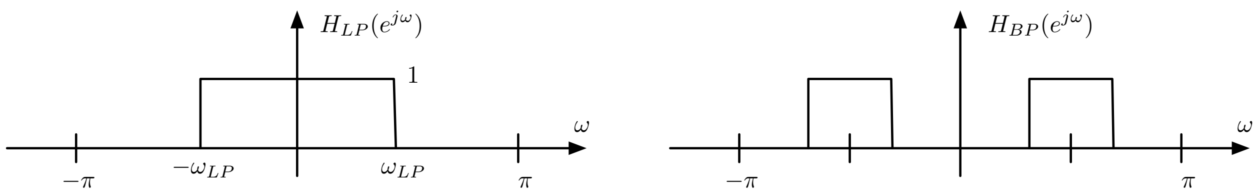


Figure 5a. Ideal lowpass and bandpass filters.

(a) Let us begin by considering ideal filters. Figure 5a above depicts the transfer functions of an ideal discrete-time lowpass filter and an ideal bandpass filter. The ideal lowpass filter has a gain of 1 in the passband and a cutoff frequency of ω_{LP}. Sketch and dimension the magnitude of the ideal bandpass filter, assuming that the bandpass filter is obtained from the lowpass filter using the relationship (again)

$$h_{BP}[n] = h_{LP}[n] 2\cos(\omega_c n)$$

Be sure to include in your sketch a numerical value for the passband gain of the bandpass filter and expressions for the center and edge frequencies of the bandpass filter in terms of the parameters ω_c and ω_{LP}. (b) Now we will consider the corresponding practical lowpass and bandpass filters. Figure 5b above depicts the edge frequencies and gain tolerances of practical filters. Note the definitions of the edge frequencies of the lowpass filter, ω_p and ω_s, and the edge frequencies of the bandpass filter, ω_{s1}, ω_{p1}, ω_{p2}, and ω_{s2}. Note also that passband tolerance is the same for the lowpass and bandpass filters (*i.e.* between $1 + \delta_p$ and $1 - \delta_p$) but the stopband ripple size is twice as large ($2\delta_s$) for the bandpass filter as it is for the lowpass filter. (This is because the ripples of the lowpass filter can add constructively when we multiply the lowpass sample response by the cosine.) Figure 5b is not drawn to scale.

Now, the discrete-time specifications for the bandpass filters that we actually will design are as follows:

- $|H(e^{j\omega})| \leq -60$ dB for $0 \leq |\omega| \leq 0.2\pi$
- -1 dB $\leq |H(e^{j\omega})| \leq 1$ dB for $0.25\pi \leq |\omega| \leq 0.65\pi$
- $|H(e^{j\omega})| \leq -60$ dB for $0.7\pi \leq |\omega| \leq \pi$

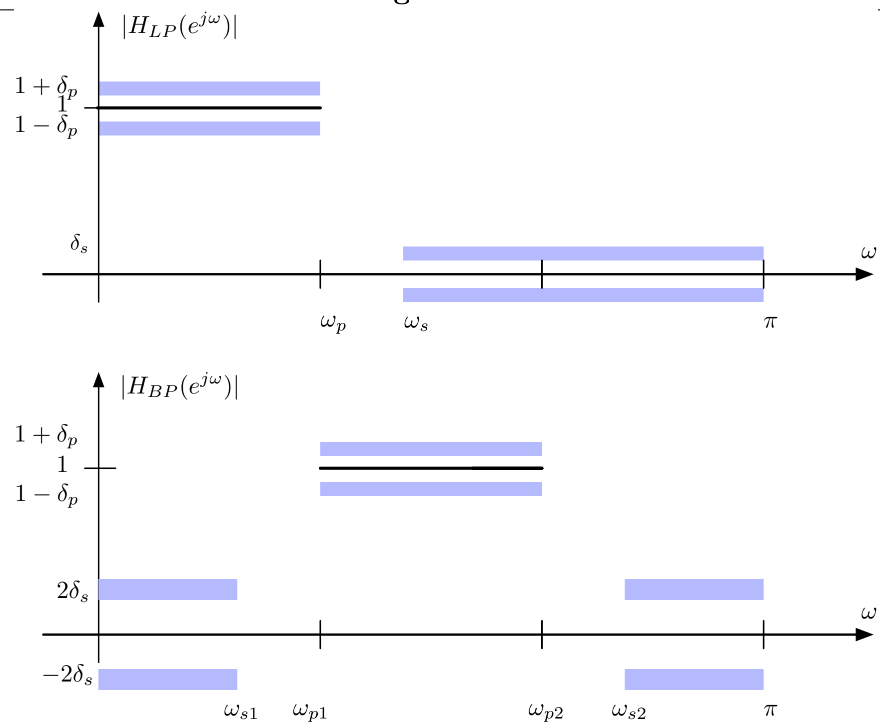


Figure 5b. Practical lowpass and bandpass filters.

1. What are the numerical values of the design parameters δ_p, δ_s, ω_{s1}, ω_{p1}, ω_{p2}, and ω_{p2} that are needed to satisfy these specfications for the *bandpass* filter?

2. What are the numerical values of the design parameters δ_p, δ_s, ω_p, and ω_s of the corresponding prototype discrete-time *lowpass filter* that are needed to meet the same specifications? Remember that the relationship between the lowpass and bandpass unit sample responses, as described in part (a), is

$$h_{BP}[n] = h_{LP}[n]2\cos(\omega_c n)$$

(c) Now let us suppose that we will design our discrete-time lowpass filter by first designing a prototype *continuous-time* filter and then transforming the design into discrete time using bilinear transformation.

1. What are the critical design parameters k_1, k_2, Ω_p, and Ω_s that we will need to use for the continuous-time filter in order to accomplish our discrete-time design using bilinear transformation?

2. If we use a Butterworth filter for our prototype (not a very efficient choice), what is the order of the filter N and the critical frequency Ω_c that we will use to design the filter? (Note that we are not asking for the actual discrete-time transfer function that results from your design.)

3. If we design the Butterworth filter according to your plans, how many multiplys per input sample will be required to accomplish the filtering? Assume that a Butterworth filter of order N produces a discrete-time filter with N poles and N zeros if we perform the conversion using bilinear transformation.

Question 6: (15%):

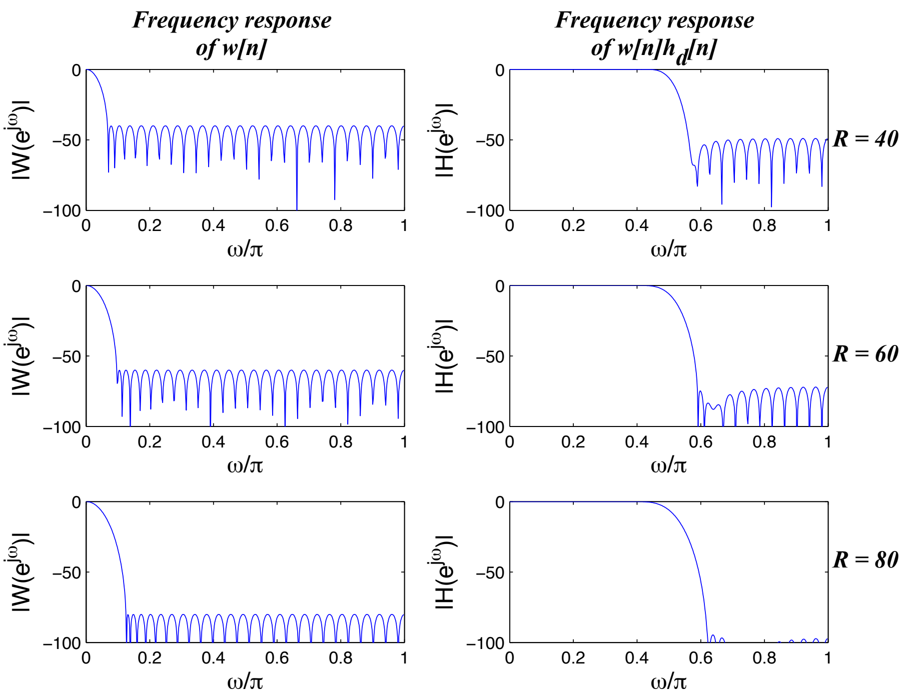


Figure 6. Examples of Dolph-Chebyschev windows for N = 51.

In this problem you will design a bandpass filter with the same specifications, but using the window design method. As a reminder, the specification of the filter in discrete time are

- $|H(e^{j\omega})| \leq -60$ dB for $0 \leq |\omega| \leq 0.2\pi$
- -1 dB $\leq |H(e^{j\omega})| \leq 1$ dB for $0.25\pi \leq |\omega| \leq 0.65\pi$
- $|H(e^{j\omega})| \leq -60$ dB for $0.7\pi \leq |\omega| \leq \pi$

As usual, you will design the filter using the window design procedure:

$$h[n] = h_d\left[n - \frac{M}{2}\right] w[n]$$

where $h_d[n]$ is a non-causal ideal filter sample response and $w[n]$ is a finite-duration window function that is nonzero for $0 \leq n \leq M$.

We would like to design this filter in linear-phase FIR form, using the window method with a window type that has not been discussed in class, the Dolph-Chebyschev window (which can be implemented using the MATLAB command `chebwin`). Figure 6 on the previous page depicts the frequency responses of several implementations of the Dolph-Chebyschev window, with three specific values of the design parameter R: 40, 60, and 80. The left column depicts the magnitude of the Fourier transform of the window itself. The right column depicts the frequency response of an ideal lowpass filter with cutoff frequency $\omega_c = 0.5\pi$ using the window design method with the Dolph-Chebyschev window. Note that the value of the parameter R affects both the width of the main lobe of $W(e^{j\omega})$ and the attenuation of its sidelobes, both of which affect the behavior of the filter that is realized, $H(e^{j\omega})$. The windows depicted in Fig. 6, and the FIR filters derived from them all had 51 nonzero samples (*i.e.* $N = 51$ and $M = 50$).

(a) As usual, the window design will be obtained by the product of a delayed ideal IIR filter response $h_d[n]$ and a finite duration window function $w[n]$:

$$h[n] = h_d[n - M/2]w[n]$$

The DTFT of the ideal bandpass filter $h_d[n]$, $H_d(e^{j\omega})$, has a magnitude that is either 0 or 1 for all frequencies and has cutoff frequencies that are midway between the passband and stopband edge frequencies in the specifications.

1. Sketch the frequency response of the ideal filter $H_d(e^{j\omega})$.
2. Obtain an analytical expression for the appropriate unit sample response $h_d[n]$ of the ideal filter, assuming that the filter length N is odd.

(b) Examine carefully the functions depicted in Fig. 6. Indicate which of the three window parameters shown (40, 60, and 80) you would use to design a filter that meets the required specifications. Describe in detail the reasoning that motivated your choice. You may make reasonable assumptions and approximations, but state what they are.

(c) On the basis of the frequency responses depicted in Fig. 6, what is the size of the window you would use for your design? Remember that the windows in the figure are all of length $N = 51$. Describe in detail the reasoning that motivated your choice. You may make reasonable assumptions and approximations, but state what they are.

(d) Now write the unit sample response of the complete FIR filter, $h[n]$. You may write the window response simply as $w[n]$ and assume that it is naturally defined over the interval $0 \leq n \leq N - 1$.

(e) How many complex multiplys per input sample will be incurred if the computation is performed using fast Fourier transforms (using either the overlap-add or overlap-save algorithm) with a DFT size of 4096? You can make reasonable assumptions, but state the assumptions that you make. Compare your answer to the corresponding answer to Question 5.

Question 7: (5%):

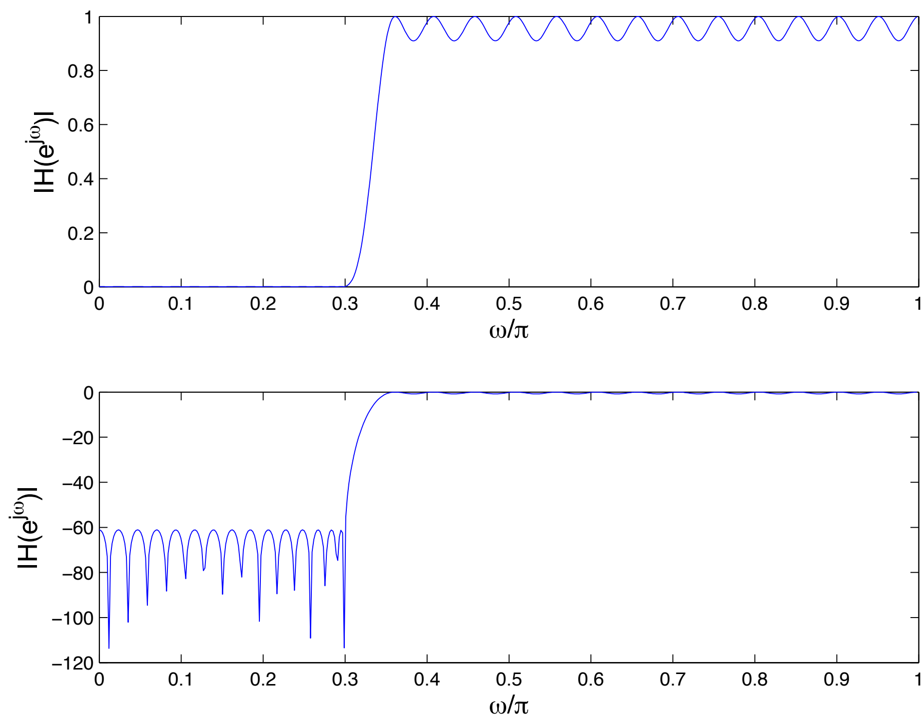


Figure 7. Equiripple implementation of the highpass filter.
Upper panel: direct magitude of the filter. Lower panel: magnitude of the filter in dB.

Figure 7 shows the frequency response of an implementation of a highpass filter using the Parks-McClellan method. Note that the same frequency response is plotted twice: as the magnitude directly (upper panel) and in decibels (lower panel).

What is your best estimate of N, the length of the filter? Keep in mind that because this is a highpass filter, the filter must be Type I. You must explain your reasoning to receive full credit (as usual).

Name: ________________________________

Solutions to Question 2:

	Sys 1	**Sys 2**	**Sys 3**	**Sys 4**	**Sys 5**	**Sys 6**	**Sys 7**	**Sys 8**
Real $h[n]$?								
Causal?								
Stable?								
Memoriless?								
Allpass?								
Linear phase?								
Minimum phase?								