Carnegie Mellon DSP Electrical & Computer ENGINEERING

Digital Signal Processing (18-491/18-691)
Spring Semester, 2025

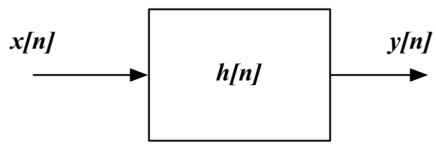
QUIZ 1

110 minutes | **Open book**
February 26, 2025 | **One sheet of notes**

We list below the ground rules governing the quiz:

- The quiz is closed book, although you may consult one page of notes, written on both sides. In addition, we will provide the tables of transforms and their properties from Chapters 2 and 3 of OSYP.
- Calculators are neither permitted nor useful for this exam.
- You will be given 10 sheets of blank printer paper, and you can use as many additional sheets as you need (just ask for them). Since your answers will be scanned, please write all answers in black ink, on both sides of the paper. Please write you name or initials on each sheet of paper and number the sheets in the order that you would like us to read them.

Reminder: Be sure to think about ways of answering the questions using reasoning, transform properties, etc., before resorting to brute-force solutions. Please note that the questions are ordered more or less in the sequence that the topics appeared in the course, and they may not be in order of difficulty. In addition, to the greatest extent possible subparts of the questions will be graded independently based on your prior answers to the question.



Question 1: (30%):

A stable LSI system is characterized by the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2}z^{-1}}{1 - 2z^{-1}}$$

(a) Obtain an analytical expression for $h[n]$, the unit sample response of the system using any method.

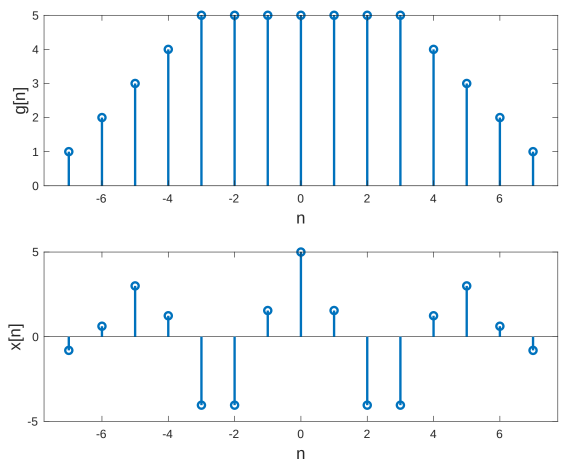
(b) Write the difference equation that realizes the transfer function. You need not specify the initial conditions, but remember that the system must be at initial rest when the input is first applied.

(c) Using any method, obtain a closed-form analytical expression for the output of the system when the system's input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n]$$

(d) Obtain an expression for the magnitude of the transfer function $H(e^{j\omega})$ for all frequencies.

(e) Sketch and dimension the phase of $H(e^{j\omega})$ for $-\pi \leq \omega \leq \pi$.



Question 2: (30%):

Short-duration pure tones are frequently used as signals in psychoacoustical experiments. In order to avoid very rapid onsets and offsets (which are perceived as distracting clicks), it is typical to gate the signals on and off gradually.

The figure above shows a cosine that is multiplied by a trapezoidally-shaped gating function $g[n]$.[1] Specifically, the signal depicted in the lower panel, $x[n]$, is described by the equation

$$x[n] = g[n]\cos(0.4\pi n)$$

The gating function $g[n]$ is defined as follows:

$$g_1[n] = \begin{cases} n+8, & -7 \leq n \leq -3 \\ 5, & -2 \leq n \leq 2 \\ 8-n, & 3 \leq n \leq 7 \\ 0, & \text{otherwise} \end{cases}$$

(a) The gating function $g[n]$ can be thought of as the convolution of two finite-duration pulses:

$$g[n] = g_1[n] * g_2[n]$$

where

$$g_1[n] = \begin{cases} 1, & -L_1 \leq n \leq L_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_2[n] = \begin{cases} 1, & -L_2 \leq n \leq L_2 \\ 0, & \text{otherwise} \end{cases}$$

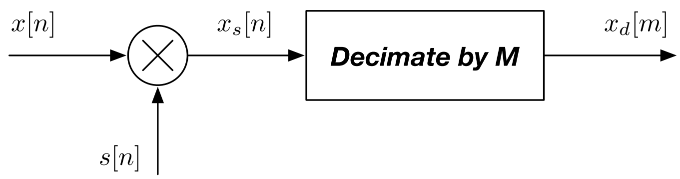
[1]This figure is a simplification to clarify the principles of the signal processing. In an actual experiment, the sampling frequency would be at least 20 kHz, the approximate duration of the tone pulse would be on the order of 200 ms, and the rise and fall times of the gating function would be on the order of 25 ms or more.

1. What are the values of the constants L_1 and L_2 that would produce the function $g[n]$ depicted in the figure above?

2. Obtain a closed form analytical expression for $G(e^{j\omega})$, the DTFT of the gating function $g[n]$.

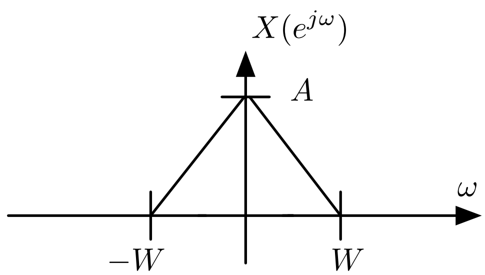
(b) The functions $g[n]$ and $x[n]$ depicted above are real and even. In real applications it is more likely that signals would be constructed to start at $n = 0$. One example would be the signal $x_2[n] = x[n-7]$. Obtain a closed-form analytical expression for the spectrum $X_2(e^{j\omega})$, the DTFT of $x_2[n]$.

Question 3: (40%):



The block diagram above describes a somewhat unconventional downsampling system. In brief, an input function $x[n]$ is multiplied by a sampling function $s[n]$ to produce the product $x_s[n]$. The function $x_s[n]$ is decimated to produce the output $x_d[m]$.

The input to the system $x[n]$ is a discrete-time signal with the DTFT $X(e^{j\omega})$ sketched below:



Please note that the signal $x[n]$ is bandlimited in that $\left|X(e^{j\omega})\right| = 0$ for $W < |\omega| \leq \pi$. The input signal $x[n]$ is multiplied by a sampling signal $s[n]$ of the form

$$s[n] = \sum_{r=-\infty}^{\infty} \delta[n-4r] - \delta[n-2-4r]$$

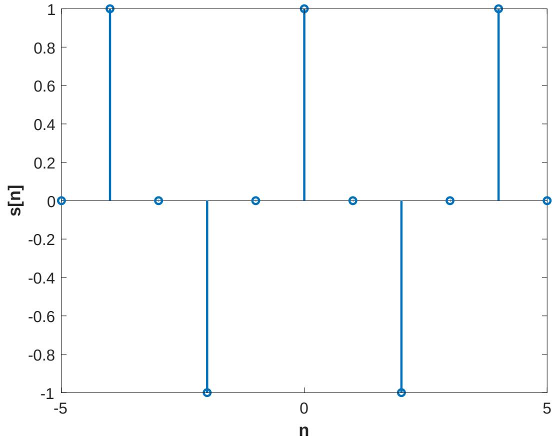
The function $s[n]$ is sketched below for a few samples of n.

(a) It can be shown that for $0 \leq \omega < 2\pi$, $S(e^{j\omega})$, the DTFT of $s[n]$, is of the form

$$S(e^{j\omega}) = a_1\delta(\omega - \pi/2) + a_3\delta(\omega - 3\pi/2)$$

Obtain values of the coefficients a_1 and a_3.

Please note: Although this question concerns the DTFT of a periodic time function, you do NOT need to make use of the general techniques described in OSYP Chapter 8 that develop the



DTFT for an arbitrary periodic time function. This problem can be solved easily by making use of your knowledge of the DTFT of the periodic sampling function used in conventional downsampling combined with the DTFT properties. Keep in mind that the function $s[n]$ is periodic.

(b) Sketch and dimension $X_s(e^{j\omega})$, the DTFT of $x_s[n]$ which, as you know, is the product of $x[n]$ and $s[n]$. You may express your answer in terms of the coefficients a_1 and a_3 if you did not complete part (a). Sketch $X_s(e^{j\omega})$ using the largest value of the bandwidth parameter W possible without causing aliasing of the spectrum in the frequency domain.

(c) What is the value of the maximum bandwidth W that does not incur aliasing?

(d) The output of the decimation box is by definition

$$x_d[m] = x_s[Mm]$$

Based on your previous work on this problem, what is the largest value of the downsampling parameter M that can be used that could enable $x[n]$ to be recovered without distortion? You must explain your reasoning to receive full credit.

(e) It is claimed that the original input $x[n]$ can be recovered by using a system that is composed of the following elements (you may not need all of them):

- Ideal sine and cosine generators
- Ideal lowpass filters
- Ideal adders and multipliers
- An ideal time expander for which the output is obtained by inserting $L-1$ zeros between each value of the input signal

Draw a block diagram of the system that will recover $x[n]$ from $x_d[m]$. You must specify the amplitudes and frequencies of all wave generators, the cutoff frequencies and gains of all filters, and the value of the parameter L of the time expander to receive full credit. There are, of course, multiple correct solutions to this part of the question, but strive to be reasonably efficient in your implementation.