

7.26. *Impulse invariance* and the *bilinear transformation* are two methods for designing discrete-time filters. Both methods transform a continuous-time system function $H_c(s)$ into a discrete-time system function $H(z)$. Answer the following questions by indicating which method(s) will yield the desired result:

- (a) A minimum-phase continuous-time system has all its poles and zeros in the left-half s -plane. If a minimum-phase continuous-time system is transformed into a discrete-time system, which method(s) will result in a minimum-phase discrete-time system?
- (b) If the continuous-time system is an all-pass system, its poles will be at locations s_k in the left-half s -plane, and its zeros will be at corresponding locations $-s_k$ in the right-half s -plane. Which design method(s) will result in an all-pass discrete-time system?
- (c) Which design method(s) will guarantee that

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}?$$

- (d) If the continuous-time system is a bandstop filter, which method(s) will result in a discrete-time bandstop filter?
- (e) Suppose that $H_1(z)$, $H_2(z)$, and $H(z)$ are transformed versions of $H_{c1}(s)$, $H_{c2}(s)$, and $H_c(s)$, respectively. Which design method(s) will guarantee that $H(z) = H_1(z)H_2(z)$ whenever $H_c(s) = H_{c1}(s)H_{c2}(s)$?
- (f) Suppose that $H_1(z)$, $H_2(z)$, and $H(z)$ are transformed versions of $H_{c1}(s)$, $H_{c2}(s)$, and $H_c(s)$, respectively. Which design method(s) will guarantee that $H(z) = H_1(z) + H_2(z)$ whenever $H_c(s) = H_{c1}(s) + H_{c2}(s)$?
- (g) Assume that two continuous-time system functions satisfy the condition

$$\frac{H_{c1}(j\Omega)}{H_{c2}(j\Omega)} = \begin{cases} e^{-j\pi/2}, & \Omega > 0, \\ e^{j\pi/2}, & \Omega < 0. \end{cases}$$

If $H_1(z)$ and $H_2(z)$ are transformed versions of $H_{c1}(s)$ and $H_{c2}(s)$, respectively, which design method(s) will result in discrete-time systems such that

$$\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \begin{cases} e^{-j\pi/2}, & 0 < \omega < \pi, \\ e^{j\pi/2}, & -\pi < \omega < 0? \end{cases}$$

(Such systems are called “90-degree phase splitters.”)

7.48. If an LTI continuous-time system has a rational system function, then its input and output satisfy an ordinary linear differential equation with constant coefficients. A standard procedure in the simulation of such systems is to use finite-difference approximations to the derivatives in the differential equations. In particular, since, for continuous differentiable functions $y_c(t)$,

$$\frac{dy_c(t)}{dt} = \lim_{T \rightarrow 0} \left[\frac{y_c(t) - y_c(t - T)}{T} \right],$$

it seems plausible that if T is “small enough,” we should obtain a good approximation if we replace $dy_c(t)/dt$ by $[y_c(t) - y_c(t - T)]/T$.

While this simple approach may be useful for simulating continuous-time systems, it is *not* generally a useful method for designing discrete-time systems for filtering applications. To understand the effect of approximating differential equations by difference equations, it is helpful to consider a specific example. Assume that the system function of a continuous-time system is

$$H_c(s) = \frac{A}{s + c},$$

where A and c are constants.

(a) Show that the input $x_c(t)$ and the output $y_c(t)$ of the system satisfy the differential equation

$$\frac{dy_c(t)}{dt} + cy_c(t) = Ax_c(t).$$

(b) Evaluate the differential equation at $t = nT$, and substitute

$$\left. \frac{dy_c(t)}{dt} \right|_{t=nT} \approx \frac{y_c(nT) - y_c(nT - T)}{T},$$

i.e., replace the first derivative by the *first backward difference*.

(c) Define $x[n] = x_c(nT)$ and $y[n] = y_c(nT)$. With this notation and the result of part (b), obtain a difference equation relating $x[n]$ and $y[n]$, and determine the system function $H(z) = Y(z)/X(z)$ of the resulting discrete-time system.

(d) Show that, for this example,

$$H(z) = H_c(s) \Big|_{s=(1-z^{-1})/T};$$

i.e., show that $H(z)$ can be obtained directly from $H_c(s)$ by the mapping

$$s = \frac{1 - z^{-1}}{T}.$$

(It can be demonstrated that if higher-order derivatives are approximated by repeated application of the first backward difference, then the result of part (d) holds for higher-order systems as well.)

- (e) For the mapping of part (d), determine the contour in the z -plane to which the $j\Omega$ -axis of the s -plane maps. Also, determine the region of the z -plane that corresponds to the left half of the s -plane. If the continuous-time system with system function $H_c(s)$ is stable, will the discrete-time system obtained by first backward difference approximation also be stable? Will the frequency response of the discrete-time system be a faithful reproduction of the frequency response of the continuous-time system? How will the stability and frequency response be affected by the choice of T ?
- (f) Assume that the first derivative is approximated by the *first forward difference*; i.e.,

$$\frac{dx(t)}{dt} \approx \frac{x(nT + T) - x(nT)}{T}$$