7.26. *Impulse invariance* and the *bilinear transformation* are two methods for designing discrete-time filters. Both methods transform a continuous-time system function $H_c(s)$ into a discrete-time system function $H(z)$. Answer the following questions by indicating which method(s) will yield the desired result:

(a) A minimum-phase continuous-time system has all its poles and zeros in the left-half s-plane. If a minimum-phase continuous-time system is transformed into a discrete-time system, which method(s) will result in a minimum-phase discrete-time system?

(b) If the continuous-time system is an all-pass system, its poles will be at locations s_k in the left-half s-plane, and its zeros will be at corresponding locations $-s_k$ in the right-half s-plane. Which design method(s) will result in an all-pass discrete-time system?

(c) Which design method(s) will guarantee that

$$H(e^{j\omega})\big|_{\omega=0} = H_c(j\Omega)\big|_{\Omega=0}?$$

(d) If the continuous-time system is a bandstop filter, which method(s) will result in a discrete-time bandstop filter?

(e) Suppose that $H_1(z)$, $H_2(z)$, and $H(z)$ are transformed versions of $H_{c1}(s)$, $H_{c2}(s)$, and $H_c(s)$, respectively. Which design method(s) will guarantee that $H(z) = H_1(z)H_2(z)$ whenever $H_c(s) = H_{c1}(s)H_{c2}(s)$?

(f) Suppose that $H_1(z)$, $H_2(z)$, and $H(z)$ are transformed versions of $H_{c1}(s)$, $H_{c2}(s)$, and $H_c(s)$, respectively. Which design method(s) will guarantee that $H(z) = H_1(z) + H_2(z)$ whenever $H_c(s) = H_{c1}(s) + H_{c2}(s)$?

(g) Assume that two continuous-time system functions satisfy the condition

$$\frac{H_{c1}(j\Omega)}{H_{c2}(j\Omega)} = \begin{cases} e^{-j\pi/2}, & \Omega > 0, \\ e^{j\pi/2}, & \Omega < 0. \end{cases}$$

If $H_1(z)$ and $H_2(z)$ are transformed versions of $H_{c1}(s)$ and $H_{c2}(s)$, respectively, which design method(s) will result in discrete-time systems such that

$$\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \begin{cases} e^{-j\pi/2}, & 0 < \omega < \pi, \\ e^{j\pi/2}, & -\pi < \omega < 0? \end{cases}$$

(Such systems are called "90-degree phase splitters.")

7.23. Consider a continuous-time system with system function

$$H_c(s) = \frac{1}{s}.$$

This system is called an *integrator*, since the output $y_c(t)$ is related to the input $x_c(t)$ by

$$y_c(t) = \int_{-\infty}^{t} x_c(\tau)d\tau.$$

Suppose a discrete-time system is obtained by applying the bilinear transformation to $H_c(s)$.

(a) What is the system function $H(z)$ of the resulting discrete-time system? What is the impulse response $h[n]$?

(b) If $x[n]$ is the input and $y[n]$ is the output of the resulting discrete-time system, write the difference equation that is satisfied by the input and output. What problems do you anticipate in implementing the discrete-time system using this difference equation?

(c) Obtain an expression for the frequency response $H(e^{j\omega})$ of the system. Sketch the magnitude and phase of the discrete-time system for $0 \leq |\omega| \leq \pi$. Compare them with the magnitude and phase of the frequency response $H_c(j\Omega)$ of the continuous-time integrator. Under what conditions could the discrete-time "integrator" be considered a good approximation to the continuous-time integrator?

Now consider a continuous-time system with system function

$$G_c(s) = s.$$

This system is a *differentiator*; i.e., the output is the derivative of the input. Suppose a discrete-time system is obtained by applying the bilinear transformation to $G_c(s)$.

(d) What is the system function $G(z)$ of the resulting discrete-time system? What is the impulse response $g[n]$?

(e) Obtain an expression for the frequency response $G(e^{j\omega})$ of the system. Sketch the magnitude and phase of the discrete-time system for $0 \leq |\omega| \leq \pi$. Compare them with the magnitude and phase of the frequency response $G_c(j\Omega)$ of the continuous-time differentiator. Under what conditions could the discrete-time "differentiator" be considered a good approximation to the continuous-time differentiator?

(f) The continuous-time integrator and differentiator are exact inverses of one another. Is the same true of the discrete-time approximations obtained by using the bilinear transformation?

7.48. If an LTI continuous-time system has a rational system function, then its input and output satisfy an ordinary linear differential equation with constant coefficients. A standard procedure in the simulation of such systems is to use finite-difference approximations to the derivatives in the differential equations. In particular, since, for continuous differentiable functions $y_c(t)$,

$$\frac{dy_c(t)}{dt} = \lim_{T\to 0}\left[\frac{y_c(t) - y_c(t-T)}{T}\right],$$

it seems plausible that if T is "small enough," we should obtain a good approximation if we replace $dy_c(t)/dt$ by $[y_c(t) - y_c(t-T)]/T$.

While this simple approach may be useful for simulating continuous-time systems, it is *not* generally a useful method for designing discrete-time systems for filtering applications. To understand the effect of approximating differential equations by difference equations, it is helpful to consider a specific example. Assume that the system function of a continuous-time system is

$$H_c(s) = \frac{A}{s+c},$$

where A and c are constants.

(a) Show that the input $x_c(t)$ and the output $y_c(t)$ of the system satisfy the differential equation

$$\frac{dy_c(t)}{dt} + cy_c(t) = Ax_c(t).$$

(b) Evaluate the differential equation at $t = nT$, and substitute

$$\left.\frac{dy_c(t)}{dt}\right|_{t=nT} \approx \frac{y_c(nT) - y_c(nT-T)}{T},$$

i.e., replace the first derivative by the *first backward difference.*

(c) Define $x[n] = x_c(nT)$ and $y[n] = y_c(nT)$. With this notation and the result of part (b), obtain a difference equation relating $x[n]$ and $y[n]$, and determine the system function $H(z) = Y(z)/X(z)$ of the resulting discrete-time system.

(d) Show that, for this example,

$$H(z) = H_c(s)\big|_{s=(1-z^{-1})/T};$$

i.e., show that $H(z)$ can be obtained directly from $H_c(s)$ by the mapping

$$s = \frac{1-z^{-1}}{T}.$$

(It can be demonstrated that if higher-order derivatives are approximated by repeated application of the first backward difference, then the result of part (d) holds for higher-order systems as well.)

(e) For the mapping of part (d), determine the contour in the z-plane to which the $j\Omega$-axis of the s-plane maps. Also, determine the region of the z-plane that corresponds to the left half of the s-plane. If the continuous-time system with system function $H_c(s)$ is stable, will the discrete-time system obtained by first backward difference approximation also be stable? Will the frequency response of the discrete-time system be a faithful reproduction of the frequency response of the continuous-time system? How will the stability and frequency response be affected by the choice of T?

(f) Assume that the first derivative is approximated by the *first forward difference;* i.e.,

$$\left.\frac{dy_c(t)}{dt}\right|_{t=nT} \approx \frac{y_c(nT+T)-y_c(nT)}{T}.$$

Determine the corresponding mapping from the s-plane to the z-plane, and repeat part (e) for this mapping.