Carnegie Mellon DSP Electrical & Computer ENGINEERING

Digital Signal Processing (18-491/18-691)
Spring Semester, 2025

Problem Set 5

Issued: 2/13/25

Due: 2/21/25 at 0100 via Gradescope

Reminder: Quiz 1 will be in class on Wednesday, February 26, during regular class hours.

We will review Quiz 1 from a previous year during the recitations on the Friday before the exam. There will also be a general review session before the exam, most likely to be scheduled during the Monday evening preceding the exam or the Sunday afternoon before then. Final time and place TBA. Most of the time during the review session will be spent answering questions and working problems .... come with questions or the review will be short and boring! The exam will cover through this problem set, or in other words the material we have discussed in OSYP Chapters 1-5, along with associated notes that have been passed out in class.

We list below the ground rules governing the quiz:

- The exam is closed book. You will be provided a set up tables of transforms and their properties which will be circulated prior to the exam. In addition you may (and should) prepare one additional sheet of notes (A4 or US letter size) written on both sides, with whatever else you would like to have in front of you. Calculators will be permitted but useless.
- The exam will be timed as usual. The nominal exam time is from 2:00 to 3:50 EST. Rather than the traditional green books, you will be asked to write your answers on plain white paper that we will provide, one side only. Put your name and a page number on every sheet. Please write in black ink only. After the exam your answers will be scanned and sent to me.
- If you have any questions while working the exam problems, please call my cell number (412) 916-7386. The TAs in the room or I will announce any exam corrections as they are identified.

Reading: During the past week we discussed continuous-time sampling and decimation and interpolation (aka downsampling and upsampling), following material in OSYP Secs. 4.0 through 4.6. We warmly recommend the discussion of sampling, decimation, and interpolation (Chapter 1) in the notes from 18-792 ADSP, which presents the same material in a form that more closely follows the sampling lecture of the past week. Next week we will discuss the DFS/DTFS and the DFT, following material in OSYP 8.0-8.6. We will also the differences between circular and linear convolution, including the overlap-add (OLA) and overlap-save (OLS) algorithms, which enable us to implement linear convolution using a series of circular convolutions. Subsequently we will introduce the fast Fourier transform (FFT) algorithm, following the presentation in OSYP Chapter 9. It is important that you know and understand the basic definitions of the DFS and the DFT, their properties, and the differences and relationships between linear and circular convolution.

Problem 5.1:

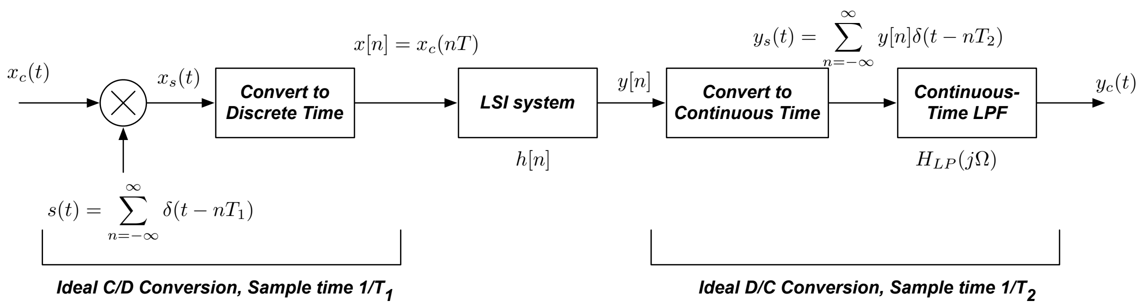


Figure 5.1-1. System for processing continuous-time signals in discrete time.

Figure 5.1-1 above is a block diagram of a generic system that converts a continuous-time signal $x_c(t)$ xinto its discrete-time representation $x[n]$ by uniform sampling, passes $h[n]$ through a discrete-time filter with sample response $h[n]$ and reconstructs the continuous-time signal $y_c(t)$ by lowpass filtering. Note that the sample times T_1 and T_2 associated with the sampling and reconstruction could be different.

Figure 5.1-2 on the next page summarizes the major steps of the sampling process. Note that the functions in the time domain are arbitrary, and that the triangular frequency representation is also arbitrary and clearly is not the shape of the actual CTFTs/DTFTs depicted. (Triangles are just easy to draw.) Also, the sampling period in the figure above is indicated by the generic T rather than T_1. Finally, note that the function $x_s(t)$ is a continuous-time function that consists of a sequence of delta functions, while the function $x[n]$ is a discrete-time sequence with the amplitudes of the discrete-time samples of $x[n]$ equal to the areas of the impulses in $x_c(t)$, and both of these are equal to the original sampled values of $x_c(t)$.

As we discussed in class, the sampling process is summarized by the equations:

$$
\begin{aligned}
x_s(t) &= x_c(t)s(t) \\
x[n] &= x_c(nT_1) \\
x_s(t) &= \sum_{n=-\infty}^{\infty} x_c(nT_1)\delta(t - nT_1) \\
x[n] &= x_c(nT_1) = \sum_{l=-\infty}^{\infty} x_c[lT_1]\delta(n - l)
\end{aligned}
$$

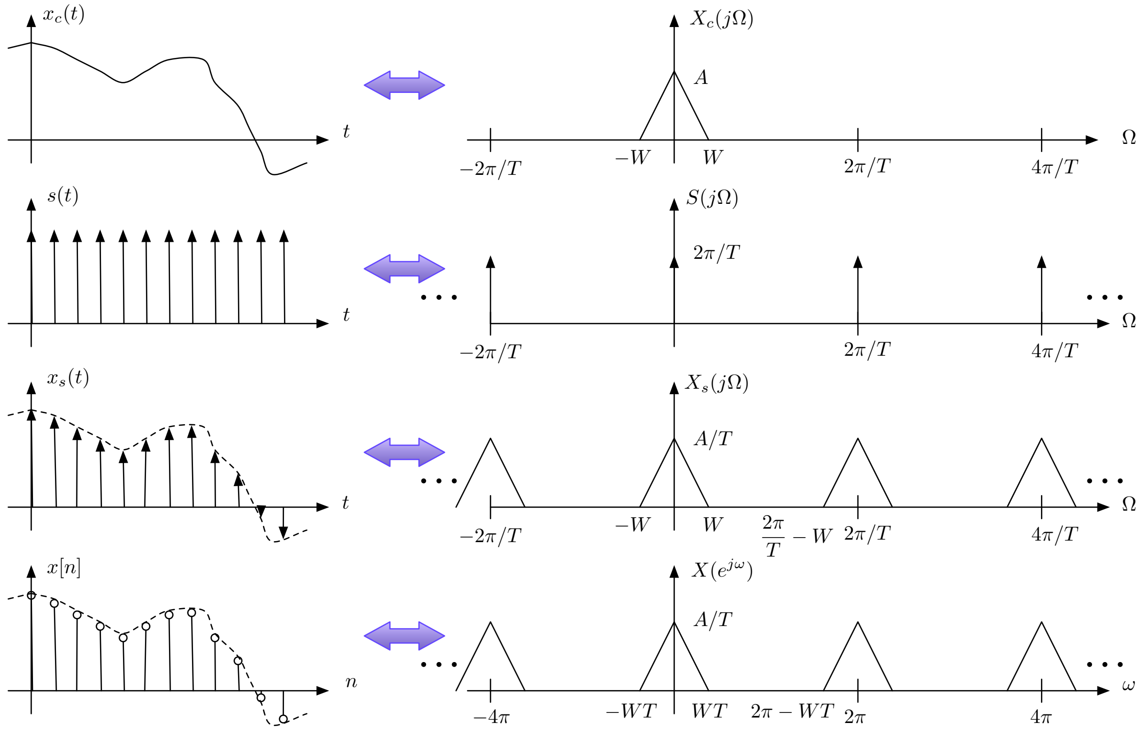


Figure 5.1-2. Time-domain and frequency-domain representations of stages of the sampling process.

and in the frequency domain,

$$S(j\Omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_1}\delta\left(\Omega - \frac{k2\pi}{T_1}\right)$$

$$X_s(j\Omega) = \frac{1}{T_1}\sum_{k=-\infty}^{\infty} X\left(j\left(\Omega - \frac{k2\pi}{T_1}\right)\right)$$

$$X(e^{j\omega}) = \frac{1}{T_1}\sum_{k=-\infty}^{\infty} X\left(j\left(\frac{\omega}{T_1} - \frac{k2\pi}{T_1}\right)\right)$$

As you know, the maximum input frequency W must be less than half the sampling rate, π/T_1 to avoid aliasing distortion.

The reconstruction of the continuous-time output begins with the discrete-time filter output converted into a continuous-time train of delta functions $y_s(t)$ separated by T_2 seconds, and with areas equal to the amplitudes of the samples of $y[n]$.

The filter $H_{LP}(j\Omega)$ is typically an ideal lowpass filter with frequency response

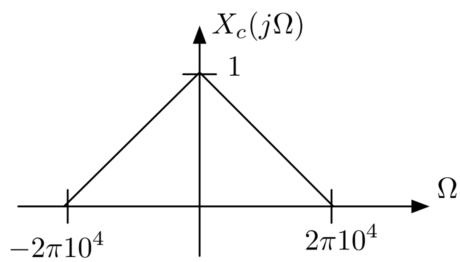
$$H(j\Omega) = \begin{cases} T_2, & |\Omega| < \frac{\pi}{T_2} \\ 0, & \text{otherwise} \end{cases}$$

Finally, assume that the discrete-time LSI system depicted above is also an ideal lowpass filter with frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.8\pi \\ 0, & \text{otherwise} \end{cases}$$

with $H(e^{j\omega})$ periodic with period 2π.

In working this problem, assume that the input signal $x_c(t)$ is bandlimited to 10 kHz, with the spectrum depicted below:



(a) Sketch and dimension in the frequency domain the functions $X_s(j\Omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, and $Y_c(j\Omega)$ for each of the following values of T_1 and T_2:

1. $T_1 = T_2 = \dfrac{1}{2.5(10^4)}$

2. $T_1 = T_2 = \dfrac{1}{1.5(10^4)}$

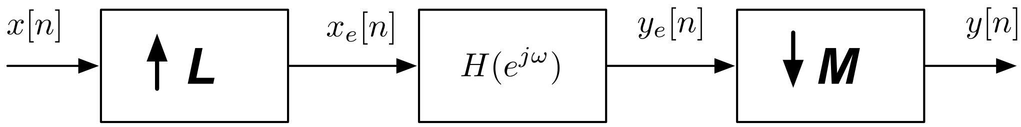
3. $T_1 = \dfrac{1}{2.5(10^4)}$ and $T_2 = \dfrac{1}{1.5(10^4)}$

(b) Now replace the input by the periodic time function $x_c(t) = \cos(2\pi 10000t)$ (*i.e.* a cosine with frequency 10 kHz). Assume for this part that the discrete-time filter is allpass with $H(e^{j\omega}) = 1$ for all ω.

Repeat part (a), subparts 1 and 2 (only) but with this cosine function as input. Also write out the time functions $y[n]$ and $y_c(t)$ that you obtain in each case.

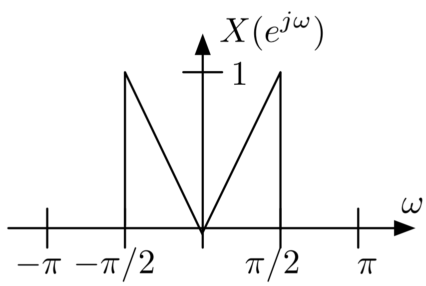
Note: It is important to remember that the frequency axis is scaled as signals are converted from $x_s(t)$ to $x[n]$ and back from $y[n]$ to $y_s(t)$. This in effect causes the delta functions to expand or contract in frequency, which affects their areas as discussed in Problem 5.1 (b) and (c). You need to be mindful of this in order to obtain the correct answers for this problem.

Problem 5.2:



Consider the multi-rate discrete-time system shown in the figure above. We know that:

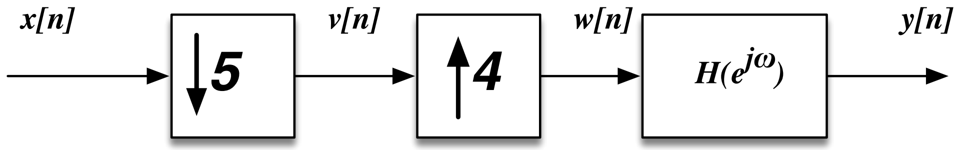
- L and M are positive integers
- $x_e[n] = x[n/L]$ for $n = rL$ and zero otherwise
- $y[n] = y_e[nM]$
- $H(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/4 \\ 0, & \pi/4 < |\omega| \leq \pi \end{cases}$



(a) Assume that $L = 2$ and $M = 4$ and that $X(e^{j\omega})$, the DTFT of $x[n]$, is real and is as shown in the figure above. Sketch and dimension the functions $X_e(e^{j\omega})$, $Y_e(e^{j\omega})$, and $Y(e^{j\omega})$, the DTFTs of $x_e[n]$, $y_e[n]$, and $y[n]$, respectively. Be sure to label clearly all important magnitudes and frequencies.

(b) Now repeat part (a) except for the input $x[n] = \cos(0.2\pi n)$. Be mindful of the fact that expanding or contracting the horizontal axis of a function in time or frequency with continuous (Dirac) delta functions will cause the areas of those delta functions to increase or decrease in area correspondingly. (See the class notes on delta functions for more info on this.)

Problem 5.3:



In the system above, the input signal is $x[n] = \sin(0.25\pi n + \pi/4)$

The system decimates the input by a factor of 5, and then upsamples the result by a factor of 4 (*i.e.* places three zeros between each successive sample of $v[n]$), and finally passes the system through the ideal lowpass filter $H(e^{j\omega})$ with frequency response

$$H(e^{j\omega}) = \begin{cases} 4, & |\omega| \le 0.3\pi \\ 0, & 0.3\pi < |\omega| \le \pi \end{cases}$$

As it turns out, this system is not particularly well designed, but do not let that bother you.

(a) Sketch and dimension the following DTFTs:

1. $V(e^{j\omega})$, the DTFT of $v[n]$
2. $W(e^{j\omega})$, the DTFT of $w[n]$
3. $Y(e^{j\omega})$, the DTFT of $y[n]$

(b) It is claimed that the output $y[n]$ can be expressed in the following form: $A\cos(\omega_0 n + \phi)$

Find numerical values for the coefficients A, ω_0, and ϕ.

Note: in obtaining the answer to this problem, please keep in mind that $\delta(at) = \frac{1}{|a|}\delta(t)$ in the distributional sense. This should be taken into consideration whenever you are working with a delta function of a continuous argument, and the scale of the horizontal axis is changed.

Problem 5.4

The function $\tilde{x}_1[n]$ is periodic with period 4, and it is equal to

$$\tilde{x}_1[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] + 3\delta[n-3] \text{ for } 0 \le n \le 3$$

This function is added to the aperiodic function

$$x_2[n] = (1/4)^n\, u[n]$$

(a) Obtain an expression for $\tilde{X}_1[k]$, the DFS coefficients of the periodic function $\tilde{x}_1[n]$.

(b) Obtain an expression (possibly including impulses) for $\tilde{X}_1(e^{j\omega})$, the DTFT of $\tilde{x}_1[n]$.

(c) Obtain an expression (possibly including impulses) for the DTFT of the sum of $\tilde{x}_1[n]$ and $x_2[n]$. Sketch the magnitude of the DTFT you obtained with respect to frequency.

MATLAB Problems

As a reminder, an easy way to handle the code part of this is to use the `publish` feature in MATLAB and submit the output .pdf to the Written assignment on gradescope.

For the MATLAB questions this week and in the future, please submit the following components of your answer to the **written** component of your homework submission on Gradescope. An easy way to handle the code part of this is to use the `publish` feature in MATLAB and submit the output .pdf to the Written assignment on gradescope.

- Answers to the written portions of the problems
- Your plots
- A pdf copy of your code

The **Matlab** component of your submission should contain only your `.m` files. We appreciate your help in complying with these formatting requests as it makes your work much easier to grade. And yes, we will deduct points for noncompliant submissions.

Problem C5.1:

In this problem, we will explore how the application of an anti-aliasing filter before downsampling can affect a downsampled signal. To accomplish this, we will utilize the MATLAB routines `decimate` and `downsample`. The function `decimate` passes the input signal through an anti-aliasing filter before downsampling, while the function `downsample` does not include an anti-aliasing filter.

You will be provided with a main file `main_5_1.m` that you must complete.

(a) [Downsampling by 2 without an anti-aliasing filter]

The wrapper file `main_5_1.m` begins with a script that generates a time-domain function that produces a rectangularly-shaped DTFT. Using this function, complete the following:

1. Compute the DTFT of the function `x` provided using either `freqz`, the function `dtft_491.m` that you developed in a problem set in a previous week, or the script for `dtft_491.m` that is provided by us. Plot the magnitude of the DTFT.
2. Now, downsample the time-domain function by a factor of 2 using the `downsample` routine. Plot the magnitude of the DTFT of the downsampled signal. Use the `hold on` keyphrase to overlay the two plots.
3. Write a few sentences describing any changes you notice between the two DTFTs. What could be the cause of the changes, if any? Is there any aliasing in this case?

(b) [Downsampling by 5 without an anti-aliasing filter]

Using the original time-domain function `x` that we provided, complete the following.

1. Repeat steps 1 and 2 from part (a), this time downsampling by a factor of 5. Again, use the `downsample` routine.
2. Write a few sentences describing any changes you notice between the two DTFTs. What could be the cause of the changes, if any? Is there any aliasing in this case?

(c) [Downsampling with an anti-aliasing filter]

Again using the original time-domain function that we provided you, complete the following.

1. Repeat steps 1 and 2 from part (a), this time downsampling by a factor of 5. This time, use the `decimate` routine instead of `downsample`; the difference being that `decimate` will pass an anti-aliasing filter over the signal before downsampling whereas `downsample` does not.
2. Write a few sentences describing any changes you notice between the two DTFTs. What could be the cause of the changes, if any? How does the DTFT of this anti-aliased downsampled signal compare to the DTFT of the downsampled signal from part (b)?

Problem C5.2: The MATLAB routines `decimate`, `interp`, and `resample` are used to accomplish decimation, interpolation, and general change of sampling rate, respectively.

You will be provided with a main file `main_5_2.m` that you must complete.

In MATLAB, enter load `mtlb`, to load a pre-recorded segment of a genuine Mathworks employee uttering the word "MATLAB". The signal is sampled at 7000 Hz.

(a) Let the sequence `x` represent samples 1000 through 1127 of the waveform `mtlb`. Plot `x` and its DTFT (perhaps using the DTFT routine that you wrote for Problem Set 2).

(b) Upsample the sequence `x` by a factor of 3. Plot the new sequence and its DTFT. Compare to your plots in (a).

(c) Downsample the original sequence `x` by a factor of 2. Plot the new sequence and its DTFT. Compare to your plots in (a) and (b).

(d) Plot the sequence that is derived from the original sequence `x` by increasing its sampling rate by a factor of 1.5. Again plot the new sequence and its DTFT.