Carnegie Mellon DSP Electrical & Computer ENGINEERING

Digital Signal Processing (18-491/18-691)
Fall Semester, 2025

Problem Set 4

Issued: 9/18/25

Due: 9/26/25 at 0100 via Gradescope.

Important note: Quiz 1 will be in class on Wednesday, October 8, during regular class hours. It will be based on material through Problem Set 5. There will be a review session before the exam, most likely to be scheduled during the Monday evening preceding the exam or the Sunday afternoon or evening before then. Final time and place TBA. Most of the time will be spent answering questions and working problems .... come with questions or the review will be short and boring! The exam will cover through this problem set, or in other words the material we have discussed in OSYP Chapters 1-5, along with associated notes that have been passed out in class.

Reading: We began our discussions of the past week or so with the computation of inverse z-transforms using the partial fraction method and linear constant-coefficient difference equations, and the development of the magnitude and phase of the DTFT from the locations of poles and zeros in the z-plane. We continued our discussion of the impact of pole/zero locations on frequency response, focusing on allpass systems, minimum and maximum-phase systems, zero-phase systems, and linear-phase systems. This material is discussed in OSYP Secs. 3.3-3.5 and 5.5-5.7, as well as the lecture notes on frequency response from pole/zero locations.

Next week we will discuss continuous–time sampling and decimation and interpolation (aka downsampling and upsampling), following material in OSYP Secs. 4.0 through 4.6. We warmly recommend the discussion of sampling, decimation, and interpolation (Chapter 1) in the notes from 18-792 ADSP, which presents the same material in a form that more closely follows the sampling lecture of the past week.

Note: We have also posted notes on how to solve difference equations analytically, how to perform inverse z-transforms using contour integration in the complex z plane, and how to calculate inverse z-transforms for systems with multiple poles in the same location. We provide notes on these techniques for your reference only, and you will not be asked questions on these techniques in your homework or for the exams.

Problem 4.1: Listed below are two z-transforms:

1. $$X_1(z) = \frac{1 - 3z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \text{ for } |z| < \frac{1}{4}$$

2. $$X_2(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.5z^{-1} + 0.125z^{-2}} \text{ for } |z| > \frac{\sqrt{2}}{4}$$

For each of the z-transforms above, determine the inverse z-transform using partial fraction expansion, as well as the first three nonzero terms of the inverse z-transform using the "long-division" method. Convince yourself that the two methods produce equivalent answers.

(You can check your answer using the transform pairs in OSYP Table 3.1, but to obtain full credit you must derive your answer using partial fractions, showing all work.)

Problem 4.2:

In Problem 1.4c and again in Problems 2.2 and 3.3 (!) we considered the convolution of the following two functions:

$$x[n] = (1/3)^{n-1}u[n-1] \text{ with } h[n] = (1/3)^{-n+1}u[-n+1]$$

As you know from the homework solutions, the result of this convolution is

$$y[n] = \begin{cases} \left(\frac{1}{8}\right)\left(\frac{1}{3}\right)^{-n}, & < 2 \\ \left(\frac{81}{8}\right)\left(\frac{1}{3}\right)^{n}, & n \geq 2 \end{cases}$$

(a) Write the z-transforms $X(z)$ and $H(z)$ for the time functions $x[n]$ and $h[n]$ along with the corresponding regions of convergence. (You already did this in Problem 3.3.)

(b) Is the system specified by $h[n]$ causal and stable? Relate your answer to the nature of the ROC.

(c) Obtain $y[n]$, the output of the system (*i.e.* the result of the convolution) using z-transform techniques with partial fraction expansion to compute the inverse z-transform. Compare your result to the answer you obtained for Problem 1.4(c), which you obtained via direct convolution. Which approach is easier?

Problem 4.3: Figure 4.3-1 below depicts the pole-zero plots for five different discrete-time LSI systems. While the axes are not labelled, the magnitudes of the pole and zero locations are all 1/4, 1/2, 0.9, 1, 1.1, and 2, while the phase angles of the pole and zero locations are 0, π, and $\pm\pi/4$.

(a) Figures 4.3-2 and 4.3-3 on the subsequent pages depict four possible transfer-function magnitudes (labelled A through D) and eight possible transfer-function phases (labeled E through L) that correspond to the LSI systems. The horizontal axis in all plots represents discrete-time frequency, plotted linearly. The vertical axis of the magnitude plots is also plotted linearly (i.e. not in decibels) with the minimum value equal to zero. The vertical axis of the phase plots is linear as well, running from $-\pi$ to π radians.

For each of the five pole-zero plots labelled System 1 through System 5, identify the corresponding magnitude and phase responses, again without using MATLAB. Please provide ample justification for your choices, as you will not receive any credit for this question without explaining your reasoning. **Note:** The magnitude plots can be multiplied by an arbitrary constant, which is why the vertical axes are not scaled.

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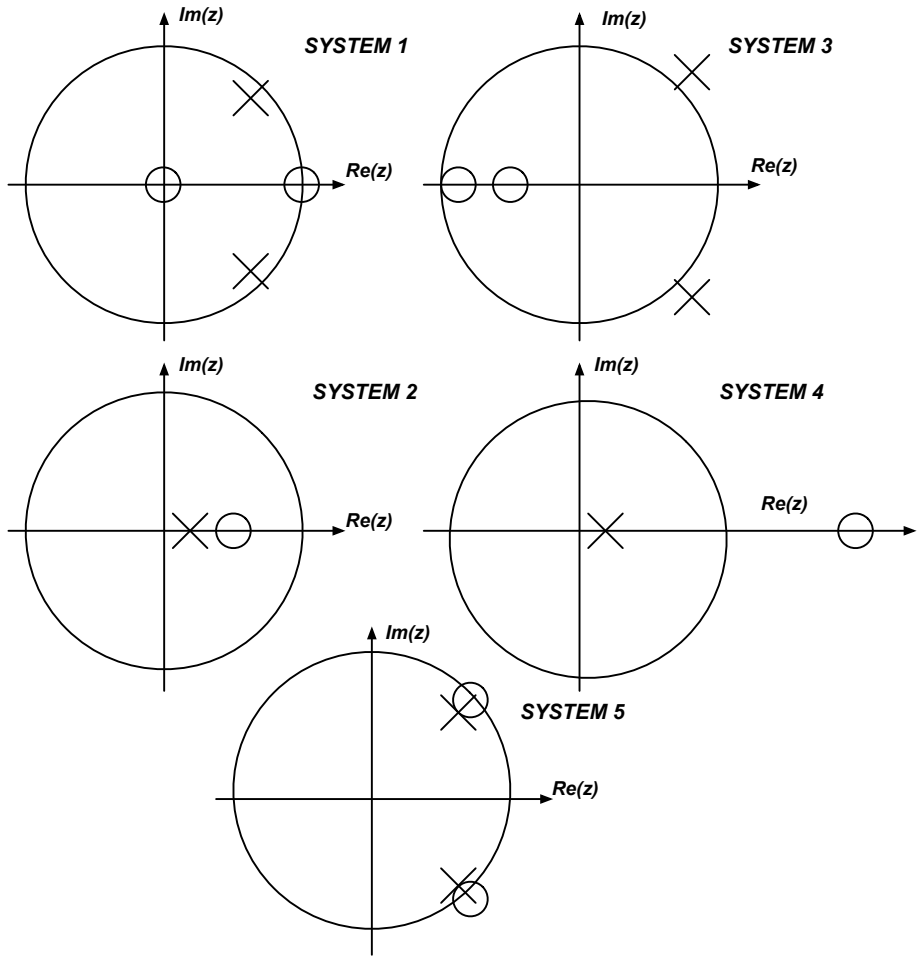

Figure 4.3-1. Pole-zero diagrams of LSI systems.

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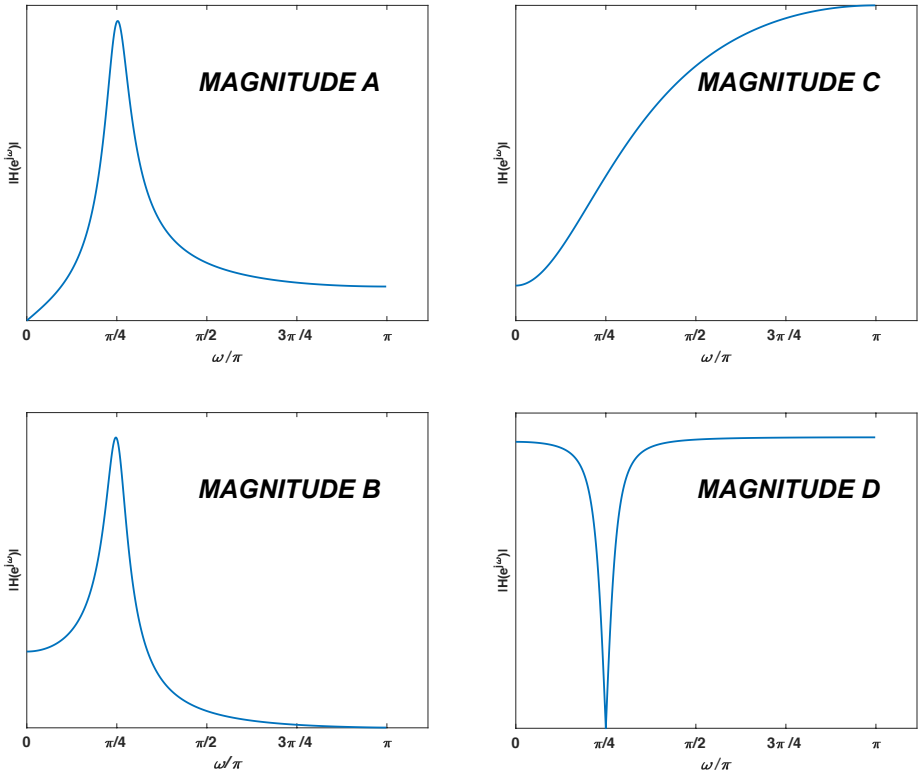

Figure 4.3-2. Potential magnitudes of frequency response of LSI systems.

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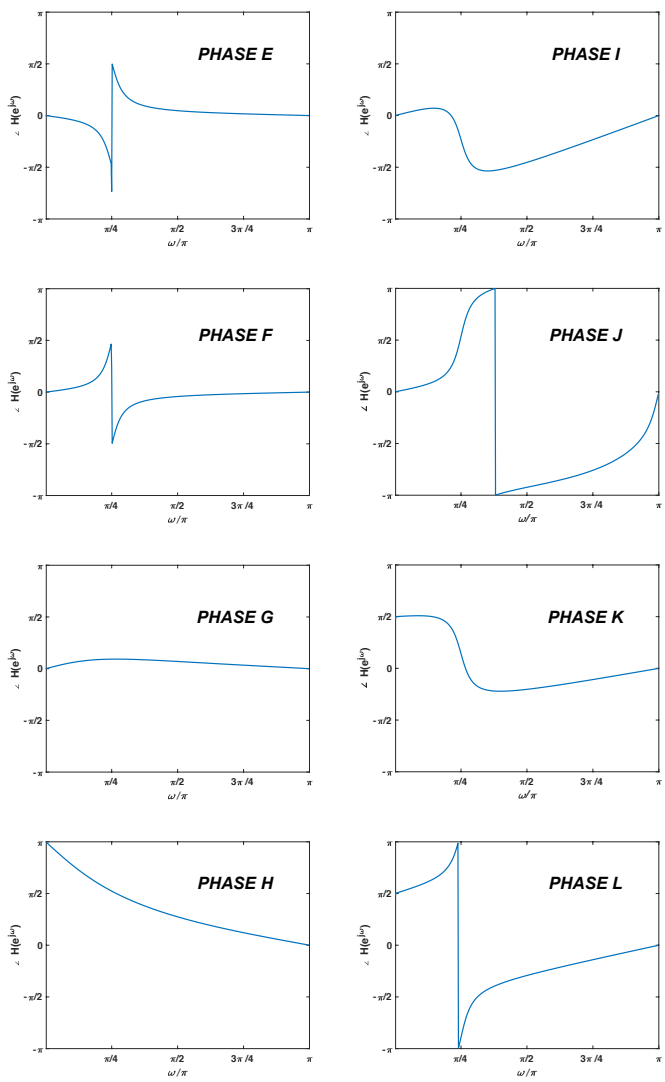

Figure 4.3-3. Potential phases of frequency response of LSI systems.

(b) Now consider the pole-zero patterns in Figure 4.3-1 once again.

1. For each of the five pole-zero patterns, sketch the pole-zero patterns of the corresponding **inverse system**. In each case, indicate whether the inverse system can be causal and stable. You must explain your answer to receive full credit.

2. For each of the five pole-zero patterns, sketch the pole-zero pattern of a second system that if cascaded with the original system would produce a composite system that is **allpass**. Indicate if the required second system does not exist, explaining your reasoning.

3. For each of the five pole-zero patterns, sketch the pole-zero pattern of a second system that if cascaded with the original system would produce a composite system that is **minimum**

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phase, without changing the magnitude of the original frequency response. Indicate if the required second system does not exist, explaining your reasoning.

4. For each of the five pole-zero patterns, sketch the pole-zero pattern of a second system that if cascaded with the original system would produce a composite system that is **linear phase**, without changing the magnitude of the original frequency response. Indicate if the required second system does not exist, explaining your reasoning.

Problem 4.4: This problem addresses an additional method of obtaining inverse z-transforms that can be helpful when working with z-transforms that are not rational ratios of polynomials in z^{-1}.

Obtain the inverse z-transform of

$$X(z) = \ln(1 - 2z), \quad |z| < \frac{1}{2}$$

(a) by using the power series

$$\ln(1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}, \quad |z| < 1,$$

(b) by first differentiating $X(z)$ and then using the derivative to recover $x[n]$

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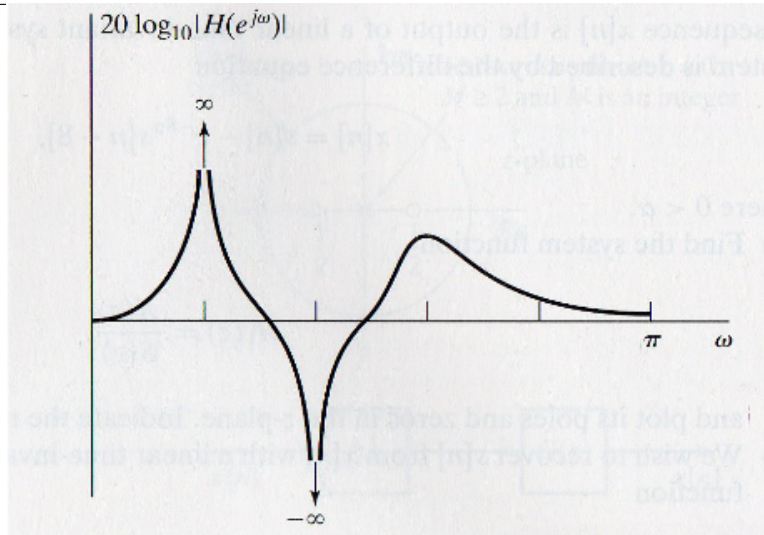

Problem 4.5:

A causal LSI system has system function $H(z)$ and areal unit sample response $h[n]$. The magnitude of $H(z)$ evaluated along the contour $z = e^{j\omega}$ is shown in the figure above.

(a) Carefully sketch a pole-zero plot for $H(z)$ showing all information about the pole and zero locations that can be inferred from he figure above.

(b) What can be said about the length of the unit sample response?

(c) Specify whether or not the system is stable. You must explain your reasoning to receive full credit.

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Problem 4.6:

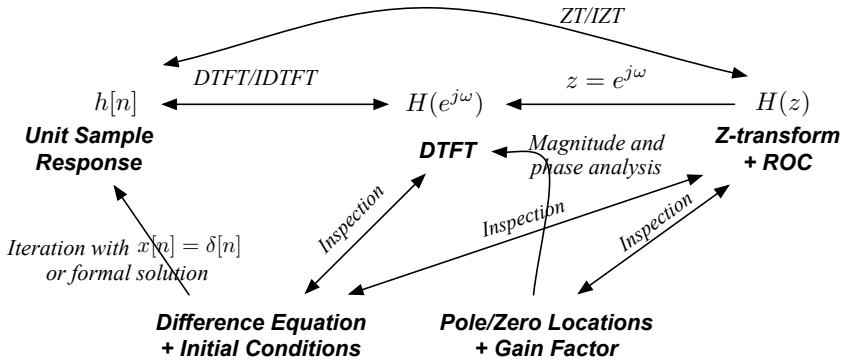

Figure 4.6-1. Summary of relationships between discrete-time representations.

Figure 4.6-1 above summarizes the five complementary representations that we have learned to characterize discrete-time signals and systems. As you know, the unit sample response is used to characterize time-domain qualities, the DTFT is the basic frequency representation, the z-transform provides greater insight into stability, causality, and an easier approach to inverse transforms, the difference equations are used to actually implement digital filters, and the pole/zero locations provide valuable design insight.

This problem is a drill that addresses these relationships between the various representations. Feel free to use MATLAB to facilitate your solutions to the problem (but be sure to tell us how you use it), unless stated otherwise.

We know the following facts about a stable LSI system under consideration:

- The system has three poles located at $z = (-1 \pm j)/2$ and $z = \frac{1}{2}$
- The system has one zero located at $z = -1$
- $$H\left(e^{j\omega}\right)\Big|_{\omega=0} = 1$$

(a) Write the z-transform $H(z)$ that represents the transfer function of the system including all multiplicative constants.

(b) Write the difference equation characterizing the system, including the initial conditions that would be required if an input were first applied to the system at $n = 0$.

(c) Determine $h[n]$, the unit sample response of the system. (Again, feel free to use MATLAB liberally here.)

(d) Is the system causal? Is it linear phase? Why or why not?

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MATLAB Problems

In working the MATLAB problems, turn in your results, a copy of the MATLAB code you developed to work the problem, as well as any additional comments you'd like to add to the MATLAB Gradescope submissions.

Reading: The MATLAB functions `roots`, `poly`, and `residuez` are especially useful in dealing with some of the issues addressed in this weeks problem set. Look over the `help` files associated with these functions. Look over the help files for more information on the other MATLAB commands cited in the problems below, as well.

For Python users: The recommended general local Python configuration is

```
- Python Version: v3.10.x
- Libraries & Recommended Versions
   - SciPy: 1.10.0
   - NumPy: 1.24.1
   - Matplotlib: 3.6.3
```

Some general recommendations for Python users include:

- Store signals as `numpy` arrays
- Read the online documentations of the functions we use to understand the differences between the MATLAB and Python implementations (although they are mostly like-for-like, nuanced differences may exist for a few of these functions)
- You would be able to find Python equivalents for most MATLAB functions by searching for `Python equivalent for <MATLAB_function_name>` online. `Stack overflow` often helps as well.

Some important reminders: You are reminded that you must submit the code for your MATLABassignments to avoid losing partial credit for your work.

Instructions for MATLAB and Python users are in separate pinned posts here (on Piazza), but the general idea is we expect a printout of code text and output, including all plots (generated as described in the posts) to be included with the rest of the written submission on Gradescope (Problem Set X submission). Code files (either `.m` or `.py`, but NOT `.mlx, .asv,` or `.ipynb`) should be submitted to the MATLAB submission on Gradescope (Problem Set X - MATLAB).

Regarding page assignments for the coding questions, there are two categories of page assignments: one that references a filename (such as "`main_2_1.m` (Published Matlab Code)") should be assigned to every page of the printout of that code file, including all outputs (eg. plots). Page assignments that reference a specific question or sub-part [such as "a) Frequency Response Plot & Written

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Response”)] should be assigned to pages containing the response a question asks, such as plots and written analysis. This system allows us to look through the entire program files and outputs for completion and correctness, as well as grade specific question sub-parts, and as with the whole assignment we may deduct points for incorrect page assignments, although well be more lenient especially with the code-file assignment ones.

For those of you using Python, we still would like you to include a published version in the written handout, and a plain-text code (.py) file for the MATLAB gradescope submission.

To generate the PDF, the only reasonably straightforward way is using `jupyter` notebooks, which you can convert to a pdf using `File -> Print` and then saving as a PDF, or using the command `jupyter nbconvert --execute --to pdf file.ipynb` (replacing `file.ipynb` with your notebook file). If youd prefer to use normal python to write your solution, use `# %%` to split cells (where you'd like a sub-parts output and graphs to show up), and use the command `jupytext --from py:percent --to notebook file.py` to convert a python file to a `jupyter` notebook.

To generate a `.py` file for the MATLAB submission (which we expect - don't submit `.ipynb` files, please), use the command `jupytext --to py file.ipynb`, which will convert your notebook into a Python file for submission.

As with MATLAB code, attach the PDF to your written submission (see that announcement for pdf merging tools), assigning pages correctly, and submit all `.py` files to the “Problem Set X - MATLAB” submission on Gradescope.

Problem C4.1: In `main_4_1.m`, Use the MATLAB function `residuez` to verify the partial fraction expansions you obtained in Problem 4.1.

For Python users: Suggested Python equivalent of `residuez`: `scipy.signal.residuez`

Problem C4.2:

In `main_4_2.m` use the MATLAB function `freqz` to verify the magnitudes and phases of the DTFTs that you obtained in Problem 4.3.

For Python users: Suggested Python equivalent of `freqz`: `scipy.signal.freqz`

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Problem C4.3: In this problem you will design a simple lowpass filter from scratch based on pole and zero locations. While your filter will need to meet a set of typical specifications, do not expect it to be the best possible filter that could be designed. We will discuss filter design procedures extensively later in the course.

The specifications of your lowpass filter will be rather casual as these things go. You must design the filter so that its transfer function satisfies the following constraints:

1. $-10 \text{ dB} < \left|H(e^{j\omega})\right| < 10 \text{ dB for } |\omega| < 0.3\pi$

2. $\left|H(e^{j\omega})\right| < -50 \text{ dB for } 0.55\pi < |\omega| \leq \pi$

To constrain the design a bit, you are required to realize your filter with only four complex poles and four complex zeros. Keep in mind that the poles and zeros must occur in complex conjugate pairs for the unit sample response to remain real.

In `main_4_3.m`, do the following:

(a) Select a set of four pole and four zero locations by trial and error that enable you to satisfy the specifications above. This is probably most easily accomplished by guessing the pole and zero locations based on your knowledge and intuition about their impact on the frequency response, converting the pole and zero locations into the numerator and denominator polynomials of the transfer function using `poly`, and then using `freqz` to verify your selection. Please use the following MATLAB statements to generate normalized plots that have the same format (which facilitates grading):

```
[h,w] = freqz(b,a);
plot(w/pi,20*log10(sqrt(10)*abs(h)/max(abs(h))));
axis([0 1 -80 10])
```

where `b` and `a` are the numerator and denominator polynomials of the transfer function, respectively. When you are satisfied with your pole and zero locations, print the frequency response curve and turn it in with your written solutions. Be sure to indicate your choice of pole and zero locations.

(b) Write the transfer function of your filter as a z-transform. You may limit the coefficients to three significant figures.

(c) Obtain a closed-form analytical expression for the impulse response of your filter using any method with the aid of MATLAB. (The command `residuez` will be very useful.) Write the equation that specifies the unit sample response, limiting your coefficients to three significant figures.

(d) Obtain the linear constant-coefficient difference equation that characterizes your filter. Write this equation out, again using three significant figures for the coefficients.

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(e) Read in the `PSO_B1short.wav` musical file that you used in Problem Set 2.[1] As in Problem C2.2, create a monophonic version of the audio file by averaging the left and right channel inputs. Using MATLAB, calculate the output of the filter when the input is the monophonic version of the audio. Use the the difference equation you developed in part (d) to obtain the output. Listen to the input and output using `soundsc` to hear the effects of the lowpass filtering.

(f) Now repeat part (e) but use the the MATLAB command `filter` to obtain the audio output. (See the `help` file for information on how to implement this.) Compare the resulting audio with your result from part (e).

Hints for Python users: Suggested Python equivalents of MATLAB functions:

```
poly -> numpy.poly
freqz -> scipy.signal.freqz
residuez -> scipy.signal.residuez
filter -> scipy.signal.lfilter
audiowrite(output.wav,y/max(abs(y)),fs) -> scipy.io.wavfile.write('output.wav',x/numpy.abs(x
```

Code to generate normalized plots:

```
import numpy as np
from matplotlib import pyplot as plt
w,h = scipy.signal.freqz(b,a);
plt.xlim(0, 1)
plt.ylim(-80 10)
plt.plot(w/np.pi,20*np.log10(np.sqrt(10)*np.abs(h)/np.abs(h).max()))
plt.show()
```

What to turn in for your MATLAB Gradescope submission: Submit your program code for all three MATLAB questions, the frequency response curve, and the audio that is output from the filter, converting it to a `.wav` file using the command

```
audiowrite('output.wav',y/max(abs(y)),fs);
```

Include with your written solutions a printed program listing generated using the MATLAB `publish` command, a printed frequency response curve, plus your answers to the questions in parts (a) through (d).

Combine all these files, along with the (trivial) code from Questions C4.1 and C4.2 into a single .zip file archive. Upload the .zip archive to the Gradescope assignment "Problem Set 4 MATLAB."

[1] The music is the opening bars from a performance of Bach's first Brandenburg concerto that I was part of many years ago.

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