Carnegie Mellon

S&S

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Department of Electrical and Computer Engineering

NOTES ON FIR FILTER DESIGN USING WINDOW FUNCTIONS

I. Introduction

These notes summarize the content of the lecture of March 18 concerning the design of FIR filters using window functions. A complete discussion of FIR window design is available in Section 7.2 of the DSP text by Oppenheim and Schafer (with Buck, hereafter known as OSB), which is available on the course Website. The MATLAB scripts used for the March 18 lecture are also available on the Web.

The purpose of these notes is to relate fill in some details about the implementations that are not in that particular section of OSB, and comment on some additional aspects of the process.

As noted in class an FIR filter is one that is described by the difference equation

$$y[n] = \sum_{l=0}^{M} b_l x[n-l]$$

and by the transfer function

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \sum_{l=0}^{M} b_l e^{-j\omega l}$$

We address the problem of designing an FIR filter that meets specifications of limited deviation from the ideal response in specified frequency bands.

The window design method does not produce filters that are optimal (in the sense of meeting the design specifications in the most computationally-efficient fashion), but the method is easy to understand and does produce filters that are reasonably good. Of all the hand-design methods, the window method is the most popular and effective.

In brief, in the window method we develop a causal linear-phase FIR filter by multipying an ideal filter that has an infinite-duration impulse response (IIR) by a finite-duration window function:

$$h[n] = h_d[n]w[n]$$

where $h[n]$ is the practical FIR filter, $h_d[n]$ is the ideal IIR prototype filter, and $w[n]$ is the finite-duration window function. An important consequence of this operation is that the

DTFTs of $h_d[n]$ and $w[n]$ undergo circular convolution in frequency:

$$H(e^{j\omega}) = \frac{1}{2\pi}\oint_{2\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

This convolution is illustrated in animated form in the MATLAB script `WindowConv.m`, which is available on the course Website.

Keep in mind that the sample response of the filter, $h[n]$, must equal zero for $n < 0$ (for the filter to be causal), must be of finite duration, and must have $h[n] = h[M-n]$ (which is the Hermitian symmetry constraint needed to obtain linear phase).

We discuss the ideal filters and the window functions in the subsequent sections.

II. Ideal filters

Ideal filters have a response that is constant in the passbands and zero in the stopbands, along with linear phase. As a reference, the transfer functions and corresponding impulse responses ideal filters are as follows:

A. Lowpass filters

$$H_d(e^{j\omega}) = \begin{cases} Ge^{-j\omega n_d}, 0 \le |\omega| \le \omega_c \\ 0, \omega_c < |\omega| \le \pi \end{cases}$$

$$h_d[n] = G\frac{\sin(\omega_c(n-n_d))}{\pi(n-n_d)}$$

B. Highpass filters

$$H_d(e^{j\omega}) = \begin{cases} Ge^{-j\omega n_d}, \omega_c \le |\omega| \le \pi \\ 0, |\omega| < \omega_c \end{cases}$$

$$h_d[n] = G\left(\delta[n-n_d] - \frac{\sin(\omega_c(n-n_d))}{\pi(n-n_d)}\right)$$

C. Bandpass filters

$$H_d(e^{j\omega}) = \begin{cases} Ge^{-j\omega n_d}, \omega_1 \le |\omega| \le \omega_2 \\ 0, |\omega| \le \omega_1 \\ 0, \omega_2 \le |\omega| \le \pi \end{cases}$$

$$h_d[n] = G\left(\frac{\sin(\omega_2(n-n_d))}{\pi(n-n_d)} - \frac{\sin(\omega_1(n-n_d))}{\pi(n-n_d)}\right) \text{ or, equivalently,}$$

$$h_d[n] = 2G\cos\left(\left(\frac{\omega_2+\omega_1}{2}\right)(n-n_d)\right)\frac{\sin\left(\left(\frac{\omega_2-\omega_1}{2}\right)(n-n_d)\right)}{\pi(n-n_d)}$$

In the filters above, the delay parameter must be an integer multiple of 1/2 for the filter to be linear phase (and must be an integer multiple of 1 in the case of the highpass filter in order for the delta function to be meaningful).

II. "Classic" window shapes

Although literally dozens of window shapes have been used for filter design, we focus on the use of the five "classic" windows discussed in OSB Sec. 7.2.1, which are all defined for $0 \le n \le M$. (Note that these are FIR windows of length $M+1$.) All of these windows are illustrated in the time domain in OSB Fig. 7.21 and in the frequency domain in OSB Fig. 7.22. The time domain and frequency responses of these windows are depicted in MATLAB scripts `WindowDemo1.m` and `WindowDemo2.m` (first half) as well.

A. Rectangular window

The rectangular window is what you would obtain if you were to simply segment a finite portion of the impulse response without any shaping in the time domain:

$$w[n] = \begin{cases} 1, 0 \le n \le M \\ 0, \text{ otherwise} \end{cases}$$

We have studied this function extensively in class, and know its DTFT to be

$$W(e^{j\omega}) = \frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}e^{-j\omega M/2}$$

Compare the plots of the original sinc function above (without the phase term) and its magnitude plotted in dB, $20\log_{10}(|W(e^{j\omega})|)$, which are shown in Fig. 1 on the next page for $M = 15$. Note that notches appear in the magnitude in dB at frequencies where there are zero crossings of the original function.

B. Bartlett (or triangular) window

The Bartlett window is triangularly shaped:

$$w[n] = \begin{cases} 1 - |(2n/M) - 1|, 0 \le n \le M \\ 0, \text{ otherwise} \end{cases}$$

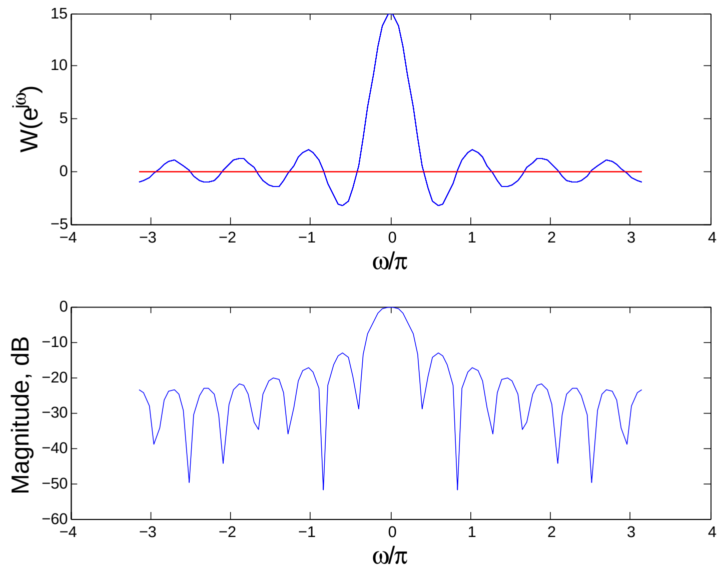

Fig. 1. Comparison of the direct representation and magnitude in dB of the discrete-time sinc function.

Because the Bartlett window can be thought of as having been obtained by convolving two rectangular windows of half the width, its transform is easily squaring the transform of the rectangular windows:

$$W(e^{j\omega}) = \left(\frac{\sin\left(\frac{M\omega}{4}\right)}{\sin\left(\frac{\omega}{2}\right)} \right)^2 e^{-j\omega M/2}$$

As you will see below, the Bartlett window has a wider mainlobe than the rectangular window, but more attenuated sidelobes.

C. Hanning window

The Hanning window(or more properly, the von Hann window) is nothing more than a raised cosine:

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M),\ 0 \le n \le M \\ 0,\ \text{otherwise} \end{cases}$$

The Hanning window has the same mainlobe width as the Bartlett window, but its sidelobes are attenuated further.

D. Hamming window

Richard W. Hamming observed that the sidelobes of the rectangular and Hanning windows are phase reversed relative to each other, so a linear combination of the two would tend to cause them to cancel each other. He searched for the linear combination that minimized the maximum sidelobe amplitude and came up with the following formulation, which represents a raised cosine on a rectangular pedestal:

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

E. Blackmam window

The Hanning and Hamming have a constant and a cosine term; the Blackman window adds a cosine at twice the frequency (you can see that this could be continued *ad infinitum*!):

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Plots of the five classic windows in the time domain are shown in Fig. 2 below.

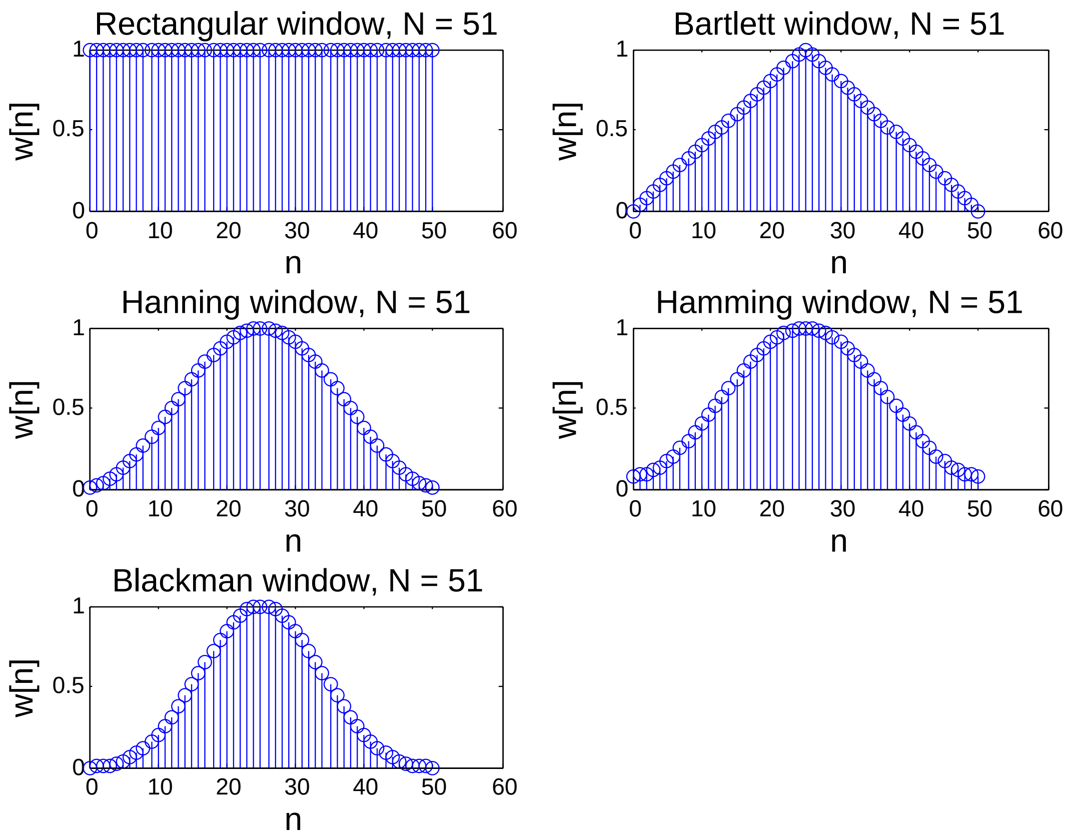

Fig. 2. "Classic" windows in the time domain.

III. The Fourier transforms of the window functions and of filters created with them

Recall that in Sec. I we noted that the frequency response of the filter we obtain is the circular convolution of the Fourier transform of the window with the frequency response of the desired ideal filter. The magnitude of the Fourier transform of the five classic windows is shown (in decibels) in Fig. 3 below.

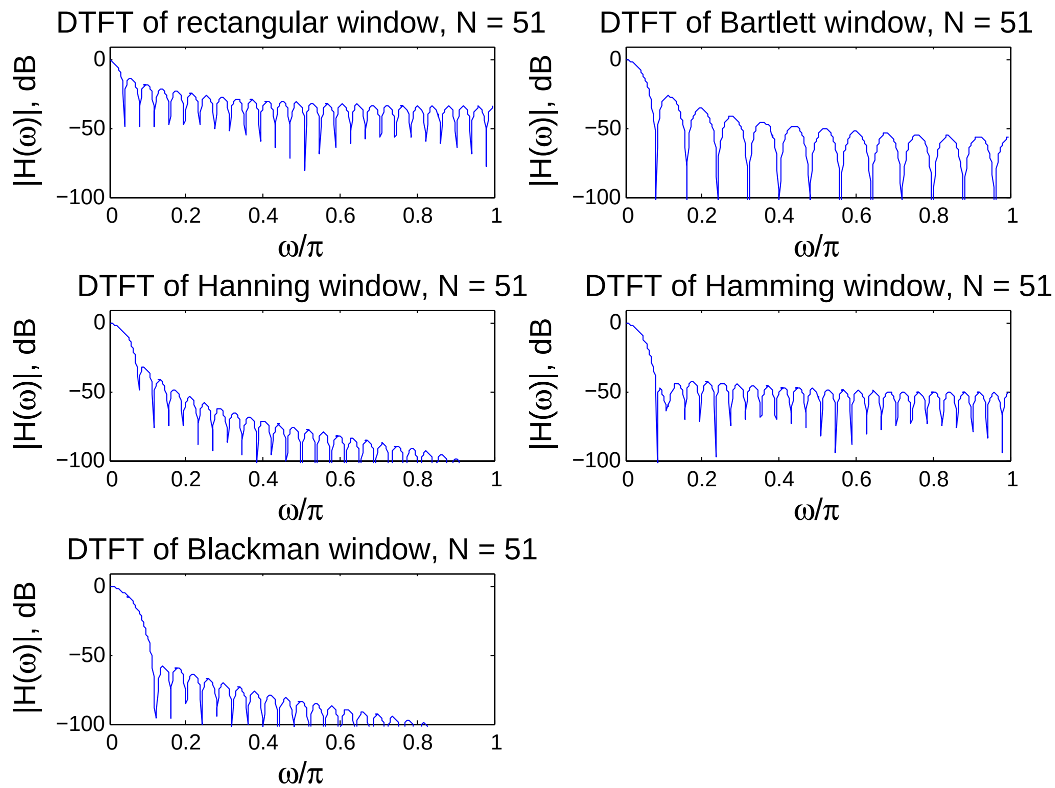

Fig. 3. Frequency response of the five classic window shapes.

The curves in Fig. 3 are valuable because they provide insight into how the windowing process impacts on the performance of the resulting non-ideal filter. As you know, the width of the transition band depends on the width of the mainlobe of the window function used to produce the FIR filter. While this width depends on the length of the window, it also depends on the window shape. It can be seen that in Fig. 3 (where all windows are of length 51), the rectangular window has the narrowest mainlobe, the Blackman window has the widest mainlobe, and the other three windows have the same (intermediate) width. The size of the ripple in the sidelobes of the window determines the relative deviation of the resulting filter gain from its ideal value. This effect is particular evident in the amount of attenuation in the "stopbands" of the resulting filter (*i.e.* those frequencies for which the ideal transfer function has zero magnitude).

The curves in Fig. 4 show the magnitude of the response of actual (nonideal) filters designed using the five windows, and an ideal lowpass prototype unit sample response with a cutoff frequency of $\omega_c = 0.4\pi$.

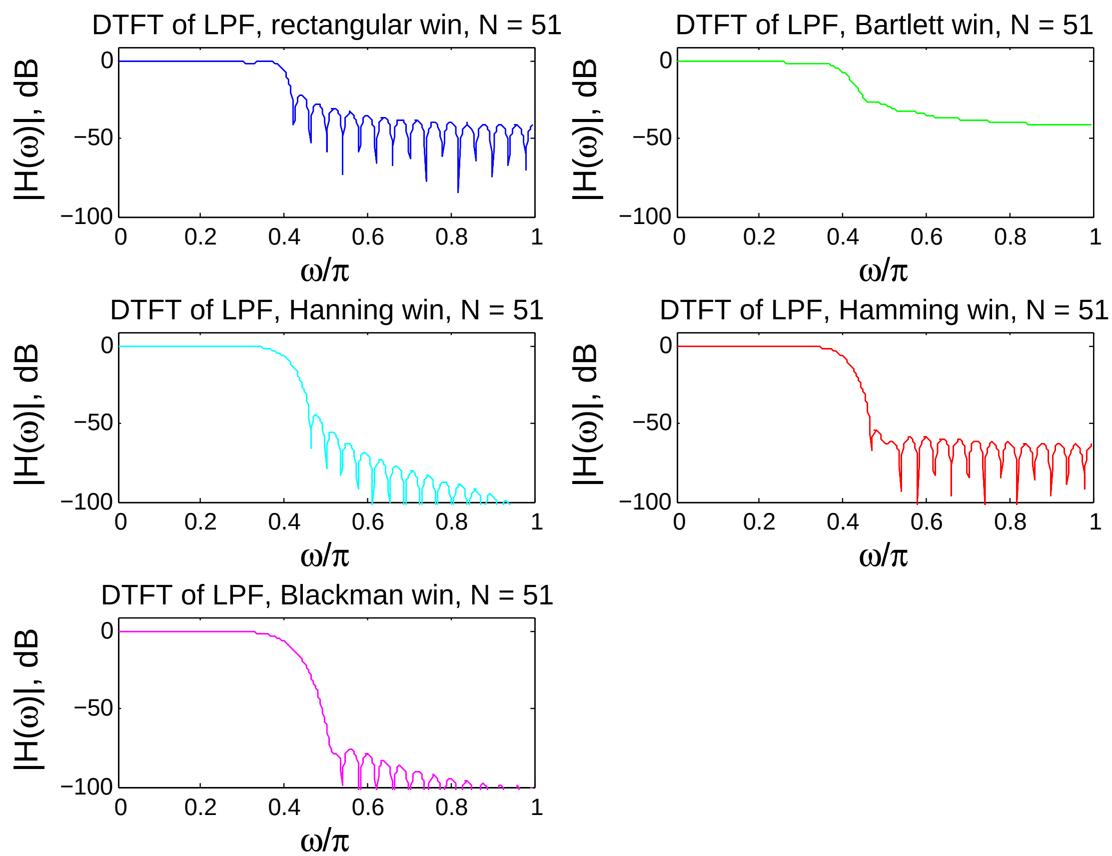

Fig. 4. Lowpass filters obtained by multiplying classic window shapes by unit impulse response of ideal lowpass filter with cutoff frequency equal to 0.4π

It can be seen that as we go from the rectangular to the Blackman windows, the stopband attenuation becomes progressively greater. It also can be seen that the transition bandwidths behave as we expected them to from the discussion above.

IV. Effect of window duration

Fig. 5 below compares the frequency response of lowpass filters of 51, 101, and 201 samples, all obtained using cutoff frequencies of 0.4π radians and Hamming windows. It can be seen that increasing the filter length decreases the transition bandwidth, but has has no effect on the stopband attenuation (provided that the window shape remains the same).

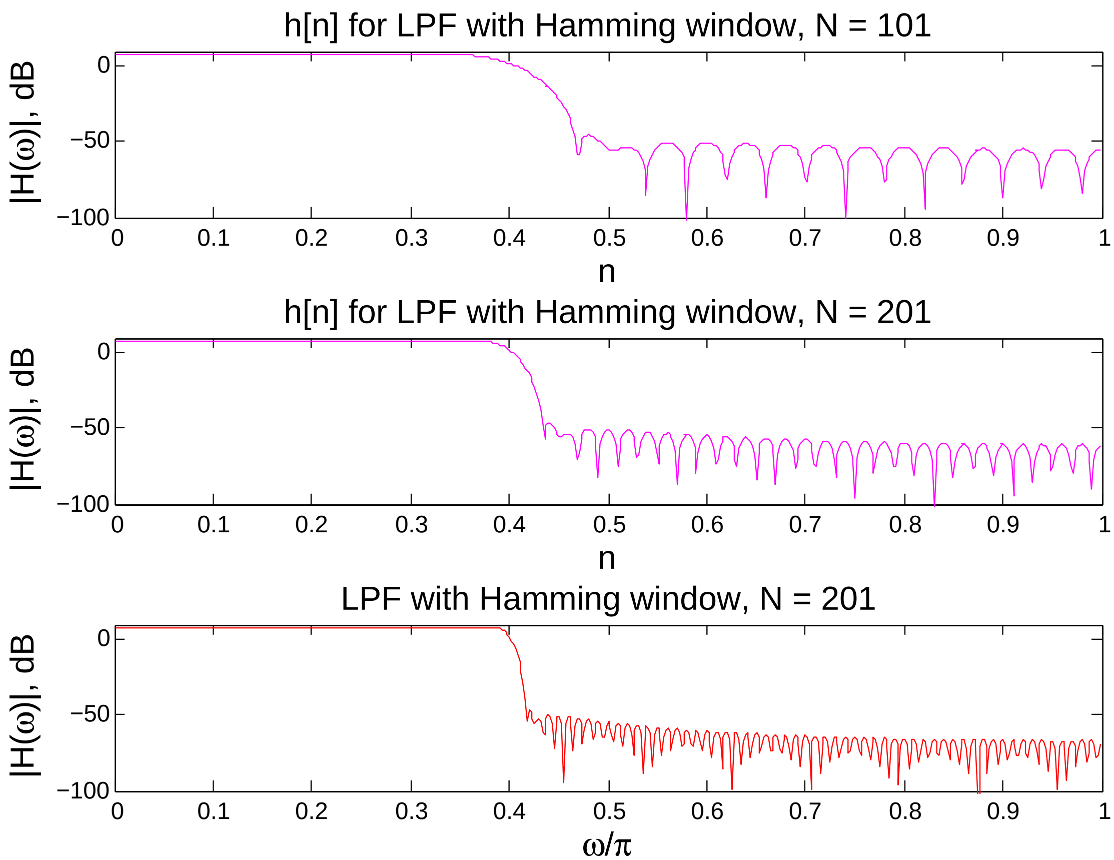

Fig. 5. Comparison of the effect of window length on the frequency response of lowpass filters. All filters have cutoff frequency of 0.4π and were designed using Hamming windows.

V. Designing FIR filters using classic window shapes

From the discussion above we have seen that the window shape affects the width of the transition band and the stopband attenuation, while the window duration affects the transition bandwidth only. This suggests a straightforward procedure that can be used to obtain an *approximate* design based on specifications. This design is facilitated by reference to OSB Table 7.1, which is partially reprinted below:

Window type	Peak Sidelobe Amplitude (Relative, dB)	Approximate Width of Mainlobe	Peak Approximation Error (dB)
Rectangular	-13	$4\pi/(M+1)$	-21
Bartlett	-25	$8\pi/M$	-25
Hanning	-31	$8\pi/M$	-44
Hamming	-41	$8\pi/M$	-53
Blackman	-57	$12\pi/M$	-74

Comparison of commonly-used windows (from OSB Table 7.1).

The numbers in the first column refer to the maxmum sidelobe amplitude of the Fourier transform of the window, can be compared to the amplitudes of the sidelobes in Fig. 3. They are not used directly in the filter design process. The expressions in the second column relate the width of the complete mainlobe of the window (including negative frequencies) to the length of the window (which actually is $M+1$). Note that the width of the mainlobe is always inversely proportional to filter length; this parameter determines the transition bandwidth of the final filter. The numbers in the third column refer to the minimum attenuation of the stopband of the filter that is designed, and can be compared to the filter shapes seen in Fig. 4.

Based on this information we can now design a filter according to an arbitrary specification. You will normally be given specifications in terms of the edges of the passband and the stopband. As an example, let us consider a lowpass filter design with the following characteristics:

- The upper edge of the passband is 0.5π radians
- The lower edge of the stopband is 0.6π radians
- The attenuation in the stopband must be at least 50 dB

On the basis of the information above, we can now design our filter:

1. **Choose the window shape to meet the stopband attenuation specifications.** In this case we would choose the Hamming window, because its minimum stopband attenuation is 53 dB.

2. **Choose the window size to meet the transition bandwidth spec, given the window shape.** In this case the transition bandwidth is 0.1π, which must equal $8\pi/M$ according to the second column of OWN Table 7.1. This means that $M = 80$ and that the filter length is 81.

3. **Obtain the filter sample response by multiplying the window by the ideal sample response.** Use an ideal filter with the cutoff frequency halfway between the passband and stopband edges. And

don't forget to delay the ideal sample response by $M/2$ samples so that the resulting filter will be linear phase!

So in this case, the filter that is developed will have the unit sample response

$$h[n] = \begin{cases} (0.54 - 0.46\cos(2\pi n/(80)))\left(\frac{\sin(0.55\pi(n-40))}{\pi(n-40)}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$