

PROBLEM SET 3**Issued:** 1/29/09**Due:** 2/4/09

Reading: The problems on this set include further properties of LTI/LSI systems, and the solution of linear constant-coefficient differential and difference equations, which is covered in OWN Sec. 2.4. Please review the lecture notes that complete the discussion of linear constant-coefficient difference equations from the Monday lecture that are being distributed. On Wednesday we began our discussion of Fourier series, which will continue next week. This material is covered in Secs. 3.2-3.7. Please do look over Sec. 3.1 for a fascinating description of the life of Fourier.

Note: Lab 1 will take place during the week of February 2.

Problem 3.1:

OWN Problem 2.43

Problem 3.2:

In this problem we will develop the complete solution to a simple first-order differential equation. Consider the LTI system with input $x(t)$ and output $y(t)$ that is defined by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \text{ where } x(t) = e^{-3t}u(t)$$

Recall that the solution to all differential equations is the sum of two solutions, the *homogenous solution* $y_h(t)$ and the *particular solution* $y_p(t)$:

$$y(t) = y_h(t) + y_p(t)$$

The homogenous solution (or natural response or transient response) reflects characteristics of the system itself, independently of the input to the system. The particular solution (or forced response or steady-state response) is dependent only on the characteristics of the input.

(a) Obtain the homogenous solution for this equation by solving the equation with the input set to zero:

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

Recall that we always assume that $y_h(t)$ is of the form Ae^{st} where A and s are coefficients to be determined. The natural frequency s is general complex. The amplitude coefficient A is usually real, but could in principle also be complex.

(b) Obtain the particular solution for this simple equation by assuming that the output is an exponential with the same time constant of the input:

$$y_p(t) = Ke^{-3t} \text{ for } t > 0$$

(c) Obtain the complete solution $y(t) = y_h(t) + y_p(t)$ by invoking the initial conditions to eliminate the remaining undertermined coefficients in your expression for $y(t)$. Recall that for the system to be LTI, we must have $y(t)$ and all of its derivatives be equal to zero right before the input is applied, i.e. at $t = 0^-$ for this input.

Problem 3.3: In this problem we will compare the result that you obtained in Problem 3.2 by formal solution of the differential equation to the solution that we would obtain for the same system using convolution.

(a) Obtain the step response $s(t)$ of this system by solving the difference equation for the unit step input $x(t) = u(t)$, again assuming initial rest. (Note that since only the input has changed, the homogeneous solution to the equation remains as before.)

(b) Since the unit impulse function $\delta(t)$ can be obtained by differentiating the unit step function (at least in the generalized or distributional sense) $\delta(t) = \frac{du(t)}{dt}$, by interchanging the linear operations of differentiation and filtering we can develop the unit impulse response by differentiating the unit sep response. What is the value of $h(t)$ for this simple system?

(c) Obtain the system output for the original input $x(t) = e^{-3t}u(t)$ by convolving that input with the unit impulse response $h(t)$ that you obtained in part (b) of this problem. Compare this result to your answer for Problem 3.2.

Which is easier for you, solution of the differential equation or evaluation of the convolution integral?

Problem 3.4: (Problems 3.4 and 3.5 repeat the computations of 3.2 and 3.3 except that they are in discrete time.)

In this problem we will develop the complete solution to a simple first-order difference equation. Consider the LTI system with input $x(t)$ and output $y(t)$ that is defined by the difference equation

$$y[n] - \left(\frac{1}{2}\right)y[n-1] = x[n] \text{ where } x[n] = \left(\frac{1}{4}\right)^n u[n]$$

Recall that the solution to all difference equations is the sum of two solutions, the *homogenous solution* $y_h[n]$ and the *particular solution* $y_p[n]$:

$$y[n] = y_h[n] + y_p[n]$$

The homogenous solution (or natural response or transient response) reflects characteristics of the system itself, independently of the input to the system. The particular solution (or forced response or steady-state response) is dependent only on the characteristics of the input.

(a) Obtain the homogenous solution for this equation by solving the equation with the input set to zero:

$$y[n] - \left(\frac{1}{2}\right)y[n-1] = 0$$

Recall that we always assume that $y_h[n]$ is of the form $A\alpha^n$ where A and α are coefficients to be determined. The natural frequency α is general complex. The amplitude coefficient A is usually real, but could in principle also be complex.

(b) Obtain the particular solution for this simple equation by assuming that the output is an exponential with the same time constant of the input:

$$x[n] = \left(\frac{1}{4}\right)^n \text{ for } n > 0$$

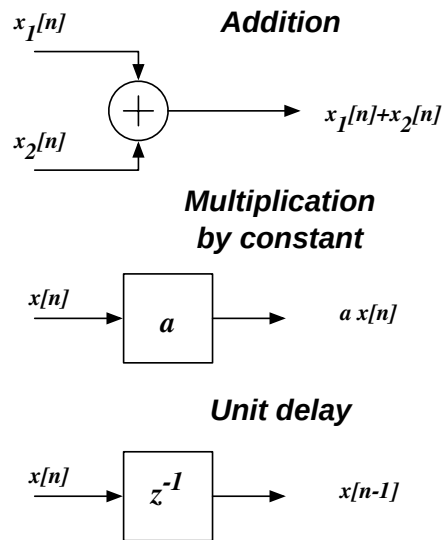
(c) Obtain the complete solution $y[n] = y_h[n] + y_p[n]$ by invoking the initial conditions to eliminate the remaining underdetermined coefficients in your expression for $y[n]$. As we discussed in the lecture, this is a bit more difficult to invoke for discrete-time systems than continuous-time systems because the time variable is discrete. In this case it is sufficient to note that $y[n] = 0$ for $n < 0$, again because this is a necessary condition for the system to be LSI.

Problem 3.5:

In Problem 3.4 you obtained the output for the LSI system characterized by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] \text{ for the input } x[n] = \left(\frac{1}{4}\right)^n, \text{ assuming initial rest.}$$

(a) Draw a block diagram of a discrete-time system that implements this difference equation using an interconnection of system elements that implement addition, multiplication by a constant, and unit delay using the elements in the figure on the next page as basic system building blocks. (The z^{-1} notation for delay will be explained and a simpler graphical representation will be developed later in the course.)



(b) In the following sections we will compare the result that you obtained in Problem 3.4 by formal solution of the difference equation to the solution that we would obtain for the same system using convolution.

1. Obtain the step response $s[n]$ of this system by solving the difference equation for the unit step input $x[n] = u[n]$, again assuming initial rest. (Note that since only the input has changed, the homogeneous solution to the equation remains as before.) You can assume a particular solution of the form

$$y_p[n] = K$$
2. Since the unit sample function can be obtained from the first difference of the unit step function, $\delta[n] = u[n] - u[n-1]$, by superposition the unit sample response can be obtained from the first difference of the unit step response, $h[n] = s[n] - s[n-1]$. What is the value of $h[n]$ for this simple system?
3. Obtain the system output for the original input $x[n] = \left(\frac{1}{4}\right)^n$ by convolving that input with the unit sample response $h[n]$ that you obtained in the previous part of this problem. Compare this result to your answer for Problem 3.4.
4. Which is easier for you, solution of the difference equation or evaluation of the convolution sum?

Problem 3.6:

Problem 3.22 in OWN, but work only subparts (d) through (e) in part (a) of the questions.