Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

Problem Set 9 (last one!)

Problem 1: Two vectors, **A** and **B** are given by:

$\mathbf{A} = 3\mathbf{a_x} + 6\mathbf{a_y} - 2\mathbf{a_z}$

$\mathbf{B} = -2\mathbf{a_x} + 5\mathbf{a_y} - 3\mathbf{a_z}$

Solution:

a) Find the dot product **A • B**

$$\begin{aligned} A \cdot B &= [3 \cdot (-2)] + [6 \cdot 5] + [(-2) \cdot (-3)] \\ &= -6 + 30 + 6 \\ &= 30 \end{aligned}$$

b) Find the angle α between **A** and **B**

$$|A| = \sqrt{3^2 + 6^2 + (-2)^2} = 7$$

$$|B| = \sqrt{(-2)^2 + 5^2 + (-3)^2} = \sqrt{38}$$

$$\cos\alpha = \frac{A \cdot B}{|A| \cdot |B|} = \frac{30}{7 \cdot \sqrt{38}} = 0.6952$$

$$\alpha = 0.80205 \quad rad$$

c) Find the cross product **A x B**

$$A \times B = \begin{vmatrix} a_x & a_y & a_z \\ 3 & 6 & -2 \\ -2 & 5 & -3 \end{vmatrix} = a_x[-18+10] + a_y[4+9] + a_z[15+12]$$

$$= -8a_x + 13a_y + 27a_z$$

d) Show that the magnitude of the cross product you found in part c) satisfies:
|A x B| = **|A|** **|B|** sin(α)

$$|A \times B| = \sqrt{(-8)^2 + 13^2 + 27^2} = 31.0161$$

$$|A||B|\sin\alpha = 7 \cdot \sqrt{38} \cdot \sin(0.80205) = 31.0161$$

Thus, the magnitude of the vector cross-product is the product of the magnitudes of the vectors times the sine of the angle between the two vectors.

Problem 2: A scalar field, U(x, y, z) is given in rectangular coordinates by:

$$U(x,y,z) = 2\frac{x}{a} + \frac{y^2}{b^2} - \frac{z^3}{c^3}$$

Where *a*, b, and c, are known constant distances.

Solution:

a)

$$\frac{\partial U}{\partial x} = \frac{2}{a}, \qquad \frac{\partial U}{\partial y} = \frac{2y}{b^2}, \qquad \frac{\partial U}{\partial z} = -\frac{3z^2}{c^3}$$

$$grad\, U = \nabla U = a_x \frac{\partial U}{\partial x} + a_y \frac{\partial U}{\partial y} + a_z \frac{\partial U}{\partial z}$$

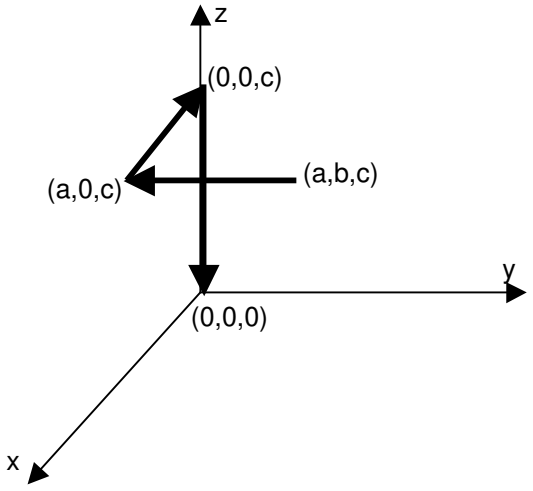
$$= a_x(\frac{2}{a}) + a_y(\frac{2y}{b^2}) + a_z(-\frac{3z^2}{c^3})$$

Physical significance ofthe gradient: Consider an example of a scalar field, temperature at different points in a room. The variation of the temperature at this point can be understood by looking at the slope of the temperature along different directions. The gradient represents the maximum slope. Thus, it gives the magnitude of the slope and the direction of the maximum slope. No wonder, gradient is a vector quantity.

b) Calculate the line integral:

$$\int_{(a,b,c)}^{(0,0,0)} \nabla U \bullet \mathbf{dl}$$

For the path shown in the figure below. The path starts at point (*a,b,c*) and follows a straight line back to point (*a*,0,c), then follows a straight line back to point (0,0,c), then follows a straight line down to point (0,0,0).



$$\int_{(a,b,c)}^{(0,0,0)} \nabla U \cdot \overline{dl} = \int_{(a,b,c)}^{(a,0,c)} \nabla U \cdot \overline{dl} + \int_{(a,0,c)}^{(0,0,c)} \nabla U \cdot \overline{dl} + \int_{(0,0,c)}^{(0,0,0)} \nabla U \cdot \overline{dl}$$

$$= \int_{(a,b,c)}^{(a,0,c)} \nabla U \cdot dy + \int_{(a,0,c)}^{(0,0,c)} \nabla U \cdot dx + \int_{(0,0,c)}^{(0,0,0)} \nabla U \cdot dz$$

$$= \int_b^0 \frac{2y}{b^2} dy + \int_a^0 \frac{2}{a} dx + \int_c^0 (-\frac{3z^2}{c^3}) dz$$

$$= \frac{y^2}{b^2}\Big|_b^0 + \frac{2x}{a}\Big|_a^0 + (-\frac{z^3}{c^3})\Big|_c^0$$

$$= -1 + 2 \cdot (-1) + 1$$

$$= -2$$

Physical Significance: In the last part, we saw that the scalar field has different slope in different directions, which corresponds to the gradient. The slope in any particular direction is equal to the dot product of the gradient and the unit vector along the required direction. Thus in this part, we found the change in temperature in going from (a, b, c) to (0, 0, 0) as the sum of the change in temperature in going along the y-axis first, then the x-axis and finally the z-axis. In the next part, we will see that the integral indeed corresponds to the difference in temperatures of the ending and the starting points.

c) Show that your result from part b) is identical to the difference between the value of the field U at the end point of the line integral and the value of U at the beginning point of the line integral, i.e., show that:

$$\int_{(a,b,c)}^{(0,0,0)} \nabla U \bullet \mathbf{\overline{dl}} = U(0,0,0) - U(a,b,c)$$

$$U(0,0,0) = 2(\frac{0}{a}) + \frac{0^2}{b^2} - \frac{0^3}{c^3} = 0$$

$$U(a,b,c) = 2(\frac{a}{a}) + \frac{b^2}{b^2} - \frac{c^3}{c^3} = 2 + 1 - 1 = 2$$

$$U(0,0,0) - U(a,b,c) = 0 - 2 = -2$$

Comparing with the result of a), we can get:

$$\int_{(a,b,c)}^{(0,0,0)} \nabla U \cdot \overline{dl} = U(0,0,0) - U(a,b,c)$$

Physical Significance: No matter along what path I keep track of the change in temperature, the net change in temperature is just the difference of the end point temperatures.

Problem 3: A vector function of space, F(x,y,z) is given by:

$$\vec{F}(x, y, z) = \vec{\mathbf{a}}_{\mathbf{x}} \frac{3x^2 y^2 z}{a^5} + \vec{\mathbf{a}}_{\mathbf{z}} \frac{xyz^3}{a^5}$$

Where "*a*" is a known constant distance.

Solution:

a) Find the scalar function of space that is defined by:

$$div\vec{F} = \nabla \bullet \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\frac{\partial F_x}{\partial x} = \frac{6xy^2z}{a^5}, \qquad \frac{\partial F_y}{\partial y} = 0, \qquad \frac{\partial F_z}{\partial z} = \frac{3xyz^2}{a^5}$$

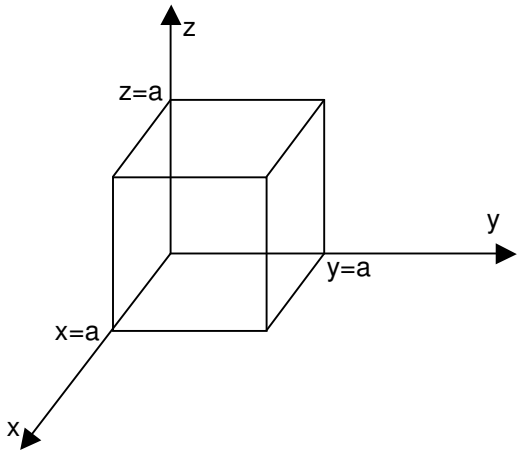
$$div\vec{F} = \nabla \bullet \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{6xy^2z + 3xyz^2}{a^5}$$

Physical significance: In problem 2, we were dealing with a scalar field, the temperature. In problems 3 and 4, we will be looking at a vector field. A good example of a vector field is the velocity field of the water flowing in the Allegheny river. At any instant of time, the water at any point can be characterized by a velocity that has a direction and a magnitude. The meaning of the divergence will become clear from the last part of this problem. Only thing to note at this point is that the divergence is a scalar quantity.

b) Calculate the volume integral:

$$\int_V \nabla \bullet \vec{F}\, dV$$

For the volume shown in the figure below. The volume is a cube with faces at x=0, x=a, y=0, y=a, z=0, and z=a.



$$\int_V \nabla \bullet \vec{F} dv = \iiint_V (\nabla \bullet \vec{F}) dxdydz$$

$$= \int_0^a \int_0^a \int_0^a (\frac{6xy^2z + 3xyz^2}{a^5}) dxdydz$$

$$= \frac{1}{a^5}\left[(\int_0^a \int_0^a \int_0^a 6xy^2z dxdydz) + (\int_0^a \int_0^a \int_0^a 3xyz^2 dxdydz) \right]$$

$$= \frac{1}{a^5}\left[(6 \cdot \frac{x^2}{2}\Big|_0^a \frac{y^3}{3}\Big|_0^a \frac{z^2}{2}\Big|_0^a) + (3 \cdot \frac{x^2}{2}\Big|_0^a \frac{y^2}{2}\Big|_0^a \frac{z^3}{3}\Big|_0^a) \right]$$

$$= \frac{1}{a^5}(\frac{a^7}{2} + \frac{a^7}{4})$$

$$= \frac{3a^2}{4}$$

c) Evaluate:

$$\vec{F} \bullet \vec{a}_n$$

i.e., the component of **F** in the direction of the outward normal (the subscript "n" denotes "normal") on *each* of the six faces of the cube (top, bottom, left, right, front, and back). Show that your result from part b) is identical to the total outward flux of **F** through the surface of the cube, i.e., show that:

$$\int_V \nabla \bullet \vec{F} dV = \oint_S \vec{F} \bullet \vec{a}_n dS$$

$$\text{on the face} \quad x = 0,\ y = 0,\ z = 0; \quad \vec{F} = 0$$

$$\text{on the face} \quad x = a; \quad \vec{a}_n = a_x \quad \vec{F} \cdot \vec{a}_n = \frac{3y^2 z}{a^3}$$

$$\text{on the face} \quad y = a; \quad \vec{a}_n = a_y \quad \vec{F} \cdot \vec{a}_n = 0$$

$$\text{on the face} \quad z = a; \quad \vec{a}_n = a_z \quad \vec{F} \cdot \vec{a}_n = \frac{xy}{a^2}$$

$$\oint_s \vec{F} \cdot \vec{a}_n ds = \iint_{x=a} \frac{3y^2 z}{a^3} dydz + \iint_{z=a} \frac{xy}{a^2} dxdy$$

$$= \frac{3}{a^3} \cdot \frac{y^3}{3}\Big|_0^a \frac{z^2}{2}\Big|_0^a + \frac{1}{a^2} \cdot \frac{x^2}{2}\Big|_0^a \frac{y^2}{2}\Big|_0^a$$

$$= \frac{a^2}{2} + \frac{a^2}{4}$$

$$= \frac{3a^2}{4}$$

As the result of b), we can get:

$$\int_V \nabla \bullet \vec{F} dv = \oint_s \vec{F} \cdot \vec{a}_n ds$$

Physical significance: This result is a very widely used theorem known as the Gauss Divergence Theorem. In the previous part, we summed over the divergence of the vector field at all points inside a closed surface. In this part, we showed that it is same as the summation of the normal component of the vector field over all points on the surface (This sum is called the flux of the vector field over the surface). The surface need not have been a regular surface. It could be arbitrary, so long as the summation is over the normal component of the vector field at every point on the surface.

To get a physical feel for the divergence, let us make use of this result. Let us shrink the extent of the closed surface to an infinitesimally small region. In the limit, the volume integral just becomes the divergence of the vector field at a point inside the infinitesimally small surface. Thus, divergence at a point represents the flux over an infinitesimally small surface enclosing it. The flux is just a number and has no direction. Hence, divergence is a scalar.

Let us see what this means in terms of the water flow. I can choose an infinitesimally small surface around a point in the water flow. I can choose the surface in such a way that the normal of the surface at any point is either parallel or anti-parallel to the velocity field. If the rate at which water enters the closed surface is exactly same as the rate at which it leaves it, then the surface integral considered will be zero and thus the divergence will be zero. When will the rate of water entering not be the same as the rate of water leaving, since the mass needs to be conserved? Never! So the velocity field of the water flow in the Allegheny river has divergence zero!! A vector field will have non-zero divergence if the point is acting as a source or a sink. In electrostatics, an electric charge is a source of the electric field and hence the divergence of the electric field is non-zero at the charge point.

Problem 4: For the same vector function **F** of space that is given in problem 3:

Solution:

a) Find the vector function of space that is defined by:

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{3x^2y^2z}{a^5} & 0 & \frac{xyz^3}{a^5} \end{vmatrix}$$

$$= \vec{a}_x[\frac{xz^3}{a^5}] + \vec{a}_y[\frac{3x^2y^2}{a^5} - \frac{yz^3}{a^5}] + \vec{a}_z[-\frac{6x^2yz}{a^5}]$$

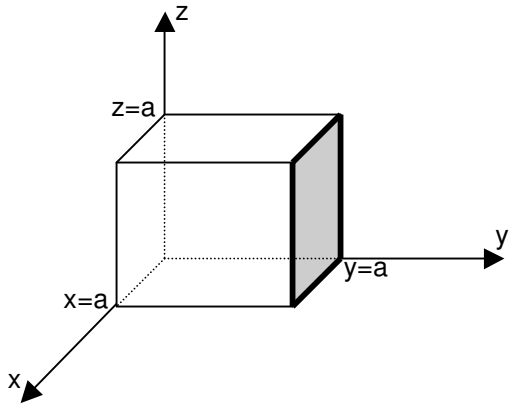
$$= \vec{a}_x[\frac{xz^3}{a^5}] + \vec{a}_y[\frac{3x^2y^2 - yz^3}{a^5}] + \vec{a}_z[-\frac{6x^2yz}{a^5}]$$

b) Consider the open surface shown in the figure on the next page. The surface is a square in the plane y=a, with edges x=0, x=a, z=0, and z=a. Evaluate:

$$\left[\nabla x\vec{F}\right] \bullet \vec{a}_n$$

i.e. the component of the *curl* **F** in the direction of the normal pointing out of the square surface in the positive "y" direction. Then, over the square, calculate this surface integral:

$$\int_S \left[\nabla x\vec{F}\right] \bullet \vec{a}_n dS$$



$$on\ the\ face\ y = a, \quad \vec{a}_n = \vec{a}_y \quad \therefore (\nabla \times \vec{F}) \cdot \vec{a}_n = \frac{3x^2}{a^3} - \frac{z^3}{a^4}$$

$$\int_s (\nabla \times \vec{F}) \cdot \vec{a}_n ds = \iint_s (\frac{3x^2}{a^3} - \frac{z^3}{a^4}) ds$$

$$= \int_0^a \int_0^a (\frac{3x^2}{a^3} - \frac{z^3}{a^4}) dxdz$$

$$= \int_0^a \int_0^a \frac{3x^2}{a^3} dxdz - \int_0^a \int_0^a \frac{z^3}{a^4} dxdz$$

$$= \frac{3}{a^3} \cdot \frac{x^3}{3}\Big|_0^a \cdot z\Big|_0^a - \frac{1}{a^4} \cdot x\Big|_0^a \cdot \frac{z^4}{4}\Big|_0^a$$

$$= a - \frac{a}{4}$$

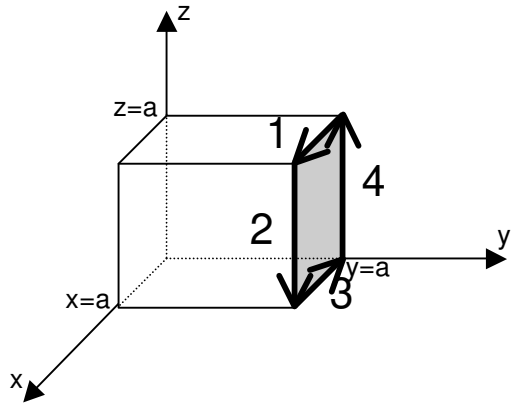
$$= \frac{3a}{4}$$

c) Calculate the line integral:

$$\oint_C \vec{F} \bullet dl$$

in the counter-clockwise direction for the closed contour that consists of the edges bound by the open square surface as viewed from the front. Show that your result from part b) is identical to the total circulation of **F** around the closed contour that bounds the open surface, i.e., show that:

$$\int_S \left[\nabla \mathrm{x} \vec{F}\right] \bullet \vec{a}_n dS = \oint_C \vec{F} \bullet dl$$



$$\oint_c \vec{F} \bullet dl = \int_1 \vec{F} \bullet dl + \int_2 \vec{F} \bullet dl + \int_3 \vec{F} \bullet dl + \int_4 \vec{F} \bullet dl$$

$on \quad line\, 1, \quad dl = dx, \quad y = a \; and \; z = a$

$$\int_1 \vec{F} \bullet dl = \int_0^a \frac{3x^2 \cdot a^2 \cdot a}{a^5} dx = \frac{3}{a^2} \cdot \frac{x^3}{3}\Big|_0^a = a$$

$on \quad line\, 2, \quad dl = dz, \quad y = a \; and \; x = a$

$$\int_2 \vec{F} \bullet dl = \int_a^0 \frac{a \cdot a \cdot z^3}{a^5} dz = \frac{1}{a^3} \cdot \frac{z^4}{4}\Big|_a^0 = -\frac{a}{4}$$

$$on \quad line\ 3, \quad dl = dx, \quad y = a\ and\ z = 0$$

$$\int_3 \vec{F} \bullet dl = 0$$

$$on \quad line\ 4, \quad dl = dz, \quad y = a\ and\ x = 0$$

$$\int_4 \vec{F} \bullet dl = 0$$

$$so \quad \oint_c \vec{F} \bullet dl = \int_1 \vec{F} \bullet dl + \int_2 \vec{F} \bullet dl + \int_3 \vec{F} \bullet dl + \int_4 \vec{F} \bullet dl$$

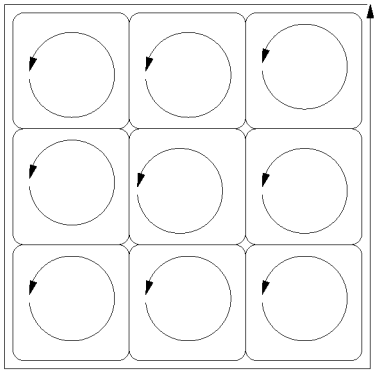
$$= a - \frac{a}{4} + 0 + 0$$

$$= \frac{3a}{4}$$

Compare with the result of b), we can get:

$$\int_s (\nabla \times \vec{F}) \cdot \vec{a}_n ds = \oint_c \vec{F} \bullet dl$$

Physical significance: This result is also a very widely used theorem known as the Stokes Theorem. In the previous part, we summed over the normal component of the curl of the vector field at all points on a surface. In this part, we showed that it is same as the summation of the tangential component of the vector field over all points on the contour bounding the surface (This sum is called the circulation of the vector field along the closed curve). The surface need not have been a plane surface. It could be arbitrarily warped, so long as the summation is over the normal component of the curl at every point on the surface. To get a physical feel for the curl, let us make use of this result. Let us shrink the extent of the contour to an infinitesimally small region. In the limit, the surface integral just becomes the curl of the vector field at a point inside the infinitesimally small contour. Thus, curl at a point represents the circulation over an infinitesimally small contour enclosing it. A classic example where one can actually see that the vector field has a non-zero circulation is a whirlpool. The vortex made by the water velocity field can be actually seen!



This figure indicates the Stokes theorem visually. The outer curve is the contour on which the line integral is being evaluated. This line integral can be thought of as the sum of the line integrals of the constitutive curves depicted by rectangles. When two rectangles share a side, the integral for that line cancels out. The number of the constitutive rectangles can be made infinitely large, so that every rectangular contour becomes infinitesimally small. Thus, each rectangular contour represents the normal component of the curl at every point. Thus, we reduce it to integral over the area of the normal component of the curl. Similar diagram can be drawn for the Gauss Divergence Theorem. It is left to the student's imagination.