Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

Problem Set 8 – solution

1. Solve the following sets of differential equations with initial conditions:

a) $dy_1(t)/dt = 3\,y_1(t) + 2\,y_2(t)$; $y_1(0) = 6$

$dy_2(t)/dt = -4\,y_1(t) - y_2(t)$; $y_2(0) = 0$

Solution:

$$y(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}; \quad \frac{dy(t)}{dt} = Ay(t); \quad y(0)=\begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$y = \hat{y}e^{\lambda t} \quad \Rightarrow \quad \frac{dy(t)}{dt} = \lambda \hat{y}e^{\lambda t} = A\hat{y}e^{\lambda t} \quad \Rightarrow \quad A\hat{y} = \lambda \hat{y}$$

Therefore this is a eigenvalue problem. First we need to find the eigenvalue and corresponding eigenvectors for the given matrix A.

$$A=\begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix}$$

$$Ax = \lambda x \quad \Rightarrow \quad \det[A-\lambda I] = 0$$

$$\Rightarrow 0 = \begin{vmatrix} 3-\lambda & 2 \\ -4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)+8 = \lambda^2 - 2\lambda + 5$$

$$\Rightarrow \lambda_1 = 1+2i, \quad \lambda_2 = 1-2i$$

$$for \quad \lambda_1 = 1+2i:$$

$$Ax_1 = \lambda_1 x_1 \quad \Rightarrow \quad [A-\lambda_1 I]x_1 = 0$$

$$\Rightarrow \begin{bmatrix} 3-(1+2i) & 2 \\ -4 & -1-(1+2i) \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-2i & 2 \\ -4 & -2-2i \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$choose \quad x_{11} = 1 \quad \Rightarrow x_{12} = -1+i, \quad x_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

$$for \quad \lambda_2 = 1-2i:$$

$$Ax_2 = \lambda_2 x_2 \quad \Rightarrow \quad [A-\lambda_2 I]x_2 = 0$$

$$\Rightarrow \begin{bmatrix} 3-(1-2i) & 2 \\ -4 & -1-(1-2i) \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+2i & 2 \\ -4 & -2+2i \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$choose \quad x_{21} = 1 \quad \Rightarrow x_{22} = -1-i, \quad x_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

Now we can give the solution to y(t),

$$y(t) = c_1 \hat{y}_1 e^{\lambda_1 t} + c_2 \hat{y}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ -1+i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{(1-2i)t}$$

$$y(0) = c_1 \begin{bmatrix} 1 \\ -1+i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 6 \\ -1+i & -1-i & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3-3i \\ 0 & 1 & 3+3i \end{bmatrix} \Rightarrow \begin{cases} c_1 = 3-3i \\ c_2 = 3+3i \end{cases}$$

$$\Rightarrow y(t) = (3-3i) \begin{bmatrix} 1 \\ -1+i \end{bmatrix} e^{(1+2i)t} + (3+3i) \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{(1-2i)t}$$

$$\Rightarrow y_1(t) = (3-3i)e^{(1+2i)t} + (3+3i)e^{(1-2i)t} = e^t (6\cos 2t + 6\sin 2t)$$

$$and \ y_2(t) = (3-3i)(-1+i)e^{(1+2i)t} + (3+3i)(-1-i)e^{(1-2i)t} = -12e^t \sin 2t$$

b) $dy_1(t)/dt = 15\, y_1(t) - 15\, y_3(t)$; $y_1(0) = 0$

$dy_2(t)/dt = -3\, y_1(t) + 6\, y_2(t) + 9y_3(t)$; $y_2(0) = -3$

$dy_3(t)/dt = 5\, y_1(t) - 5\, y_3(t)$; $y_3(0) = 4$

Solution:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}; \quad \frac{dy(t)}{dt} = Ay(t); \quad y(0) = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$$

$$y = \hat{y}e^{\lambda t} \quad \Rightarrow \quad \frac{dy(t)}{dt} = \lambda \hat{y} e^{\lambda t} = A\hat{y}e^{\lambda t} \quad \Rightarrow \quad A\hat{y} = \lambda \hat{y}$$

Therefore this is a eigenvalue problem. First we need to find the eigenvalue and corresponding eigenvectors for the given matrix A.

$$A = \begin{bmatrix} 15 & 0 & -15 \\ -3 & 6 & 9 \\ 5 & 0 & -5 \end{bmatrix}$$

$$Ax = \lambda x \quad \Rightarrow \quad \det[A - \lambda I] = 0$$

$$\Rightarrow \begin{vmatrix} 15-\lambda & 0 & -15 \\ -3 & 6-\lambda & 9 \\ 5 & 0 & -5-\lambda \end{vmatrix} = (15-\lambda)(6-\lambda)(-5-\lambda) - 5(6-\lambda)(-15) = 0$$

$$\Rightarrow \lambda(6-\lambda)(\lambda-10) = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 6, \quad \lambda_3 = 10$$

for $\lambda_1 = 0$:

$$Ax_1 = \lambda_1 x_1 \quad \Rightarrow \quad [A - \lambda_1 I]x_1 = 0$$

$$\Rightarrow \begin{bmatrix} 15 & 0 & -15 \\ -3 & 6 & 9 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

choose $x_{11} = 1 \quad \Rightarrow x_{12} = -1,$ *and* $x_{13} = 1 \quad \therefore x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda_2 = 6$:

$$Ax_2 = \lambda_2 x_2 \quad \Rightarrow \quad [A - \lambda_2 I]x_2 = 0$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & -15 \\ -3 & 0 & 9 \\ 5 & 0 & -11 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_{21} = 0 \quad$ *and* $\quad x_{23} = 0,$ *choose* $x_{22} = 1 \quad \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda_3 = 10$:

$$Ax_3 = \lambda_3 x_3 \quad \Rightarrow \quad [A - \lambda_3 I]x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 5 & 0 & -15 \\ -3 & -4 & 9 \\ 5 & 0 & -15 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_{32}=0, \quad choose\ x_{31}=3 \quad \Rightarrow \quad x_{33}=1, \quad \therefore x_3=\begin{bmatrix}3\\0\\1\end{bmatrix}$$

Now we can give the solution to y(t),

$$y(t)=c_1\hat{y}_1e^{\lambda_1 t}+c_2\hat{y}_2e^{\lambda_2 t}+c_3\hat{y}_3e^{\lambda_3 t}=c_1\begin{bmatrix}1\\-1\\1\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}e^{6t}+c_3\begin{bmatrix}3\\0\\1\end{bmatrix}e^{10t}$$

$$y(0)=c_1\begin{bmatrix}1\\-1\\1\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}3\\0\\1\end{bmatrix}=\begin{bmatrix}1&0&3\\-1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\-3\\4\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&0&3&0\\-1&1&0&-3\\1&0&1&4\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0&6\\0&1&0&3\\0&0&1&-2\end{bmatrix}\Rightarrow\begin{cases}c_1=6\\c_2=3\\c_3=-2\end{cases}$$

$$\Rightarrow y(t)=6\begin{bmatrix}1\\-1\\1\end{bmatrix}+3\begin{bmatrix}0\\1\\0\end{bmatrix}e^{6t}-2\begin{bmatrix}3\\0\\1\end{bmatrix}e^{10t}$$

$$\Rightarrow\begin{cases}y_1(t)=6-6e^{10t}\\y_2(t)=-6+3e^{6t}\\y_3(t)=6-2e^{10t}\end{cases}$$

c) $dy_1(t)/dt = 8\,y_1(t) + 3\,y_3(t)$; $y_1(0) = 15$

$dy_2(t)/dt = 2\,y_1(t) + 2\,y_2(t) + y_3(t)$; $y_2(0) = 9$

$dy_3(t)/dt = 2\,y_1(t) + 3\,y_3(t)$; $y_3(0) = -2$

Solution:

$$y(t)=\begin{bmatrix}y_1(t)\\y_2(t)\\y_3(t)\end{bmatrix}; \quad \frac{dy(t)}{dt}=Ay(t); \quad y(0)=\begin{bmatrix}15\\9\\-2\end{bmatrix}$$

$$y=\hat{y}e^{\lambda t} \quad \Rightarrow \quad \frac{dy(t)}{dt}=\lambda\hat{y}e^{\lambda t}=A\hat{y}e^{\lambda t} \quad \Rightarrow \quad A\hat{y}=\lambda\hat{y}$$

Therefore this is a eigenvalue problem. First we need to find the eigenvalue and corresponding eigenvectors for the given matrix A.

$$A=\begin{bmatrix}8&0&3\\2&2&1\\2&0&3\end{bmatrix}$$

$$Ax=\lambda x \quad \Rightarrow \quad \det[A-\lambda I]=0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & 0 & 3 \\ 2 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = (8-\lambda)(2-\lambda)(3-\lambda) - 2(2-\lambda)3 = 0$$

$$\Rightarrow (9-\lambda)(2-\lambda)^2 = 0$$

$$\Rightarrow \lambda_1 = 9, \quad \lambda_2 = \lambda_3 = 2$$

for $\lambda_1 = 9$:

$$Ax_1 = \lambda_1 x_1 \quad \Rightarrow \quad [A - \lambda_1 I]x_1 = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 3 \\ 2 & -7 & 1 \\ 2 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textit{choose } x_{11} = 3, \quad \Rightarrow x_{12} = 1 \quad \textit{and} \quad x_{13} = 1, \quad \therefore x_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = \lambda_3 = 2$:

$$Ax_{2,3} = \lambda_{2,3} x_{2,3} \quad \Rightarrow \quad [A - \lambda_{2,3} I]x_{2,3} = 0$$

$$\Rightarrow \begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{(2,3)1} \\ x_{(2,3)2} \\ x_{(2,3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(2,3)1} \\ x_{(2,3)2} \\ x_{(2,3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textit{choose } x_{21} = 0, \quad \Rightarrow x_{22} = 1 \quad \textit{and} \quad x_{23} = 0, \quad \therefore x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\textit{choose } x_{31} = 1, \quad \Rightarrow x_{32} = 0 \quad \textit{and} \quad x_{33} = -2, \quad \therefore x_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Now we can give the solution to y(t),

$$y(t) = c_1 \hat{y}_1 e^{\lambda_1 t} + c_2 \hat{y}_2 e^{\lambda_2 t} + c_3 \hat{y}_3 e^{\lambda_3 t} = c_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} e^{9t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t}$$

$$y(0)=c_1\begin{bmatrix}3\\1\\1\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\0\\-2\end{bmatrix}=\begin{bmatrix}3&0&1\\1&1&0\\1&0&-2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}15\\9\\-2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}3&0&1&15\\1&1&0&9\\1&0&-2&-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0&4\\0&1&0&5\\0&0&1&3\end{bmatrix}\Rightarrow\begin{cases}c_1=4\\c_2=5\\c_3=3\end{cases}$$

$$\Rightarrow y(t)=4\begin{bmatrix}3\\1\\1\end{bmatrix}e^{9t}+5\begin{bmatrix}0\\1\\0\end{bmatrix}e^{2t}+3\begin{bmatrix}1\\0\\-2\end{bmatrix}e^{2t}$$

$$\Rightarrow\begin{cases}y_1(t)=12e^{9t}+3e^{2t}\\y_2(t)=4e^{9t}+5e^{2t}\\y_3(t)=4e^{9t}-6e^{2t}\end{cases}$$

2. A volume charge density ρ(x, y, z) is distributed over the volume of a rectangular solid that extends between x = 0 and x = a, between y = 0 and y = b, and between z = 0 and z = c. The functional dependence of ρ(x, y, z) upon position within the volume is:

$$\rho(x,y,z) = \rho_0 \frac{(a-x)^3 (b-y)^2 z}{a^3 b^2 c}$$

where ρ_o is a known constant with units C/m^3. Find the total charge Q within the rectangular solid.

Solution:

$$\begin{aligned}
Q &= \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} \rho(x,y,z)\,dx\,dy\,dz \\
&= \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} \rho_0 \frac{(a-x)^3 (b-y)^2 z}{a^3 b^2 c}\,dx\,dy\,dz \\
&= \frac{\rho_0}{a^3 b^2 c}\int_{z=0}^{c} z\,dz \int_{y=0}^{b} (b-y)^2\,dy \int_{x=0}^{a} (a-x)^3\,dx \\
&= \frac{\rho_0}{a^3 b^2 c}\left[\frac{z^2}{2}\right]_{z=0}^{c}\left[-\frac{(b-y)^3}{3}\right]_{y=0}^{b}\left[-\frac{(a-x)^4}{4}\right]_{x=0}^{a} \\
&= \frac{\rho_0}{a^3 b^2 c}\cdot\frac{c^2}{2}\cdot\frac{b^3}{3}\cdot\frac{a^4}{4} \\
&= \rho_0 \frac{abc}{24}
\end{aligned}$$

3. A certain antenna emits a power per unit area that varies with position in the region that surrounds the antenna. In a spherical coordinates system whose origin coincides with the center of the antenna, the power density propagates radially outward, and takes the form:

$$p(r,\theta,\phi) = \frac{P_0}{r^2}\sin^4(\theta)\cos^2(\phi)$$

where P_0 is a constant with units Watts and r and θ are spherical coordinates. By integrating the power per unit area over the surface of a sphere that surrounds the antenna, find the total power P that radiates outward from the antenna.

Solution:

$$\begin{aligned}
P &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} p(r,\theta,\phi) r^2 \sin\theta\,d\theta\,d\phi \\
&= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{P_0}{r^2}\sin^4\theta\cos^2\phi\, r^2 \sin\theta\,d\theta\,d\phi\,, \\
&= P_0 \int_{\theta=0}^{\pi} \sin^5\theta\,d\theta \int_{\phi=0}^{2\pi} \cos^2\phi\,d\phi \\
&= P_0\left\{-\frac{\sin^4\theta\cos\theta}{5} + \frac{4}{5}\left[-\frac{1}{3}\cos\theta(\sin^2\theta+2)\right]\right\}_{\theta=0}^{\pi}\left[\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right]_{\phi=0}^{2\pi} \\
&= P_0 \frac{4}{5}\cdot\frac{4}{3}\cdot\frac{1}{2}\cdot 2\pi = \frac{16}{15}\pi P_0
\end{aligned}$$

4. A surface charge density σ(ϕ, z) is distributed over the side-wall of a circular cylinder whose axis is the z-axis, has radius R, and extends between z = 0 and z = H. The functional dependence of σ(ϕ, z) upon angular position and height on the surface is:

$$\sigma(\phi, z) = \sigma_0 \frac{z^2}{H^2} \cos^2 \phi$$

where σ_o is a known constant with units C/m^2. Find the total charge Q on the side wall of the cylinder.

Solution:

$$\begin{aligned} Q &= \int_{z=0}^{H} \int_{\phi=0}^{2\pi} \sigma(\phi, z) R \, d\phi \, dz \\ &= \int_{z=0}^{H} \int_{\phi=0}^{2\pi} \sigma_0 \frac{z^2}{H^2} \cos^2 \phi \, R \, d\phi \, dz \\ &= \sigma_0 \frac{R}{H^2} \int_{z=0}^{H} z^2 dz \int_{\phi=0}^{2\pi} \cos^2 \phi d\phi \\ &= \sigma_0 \frac{R}{H^2} \left[\frac{z^3}{3} \right]_{z=0}^{H} \left[\frac{1}{2}\phi + \frac{1}{4} \sin 2\phi \right]_{\phi=0}^{2\pi} \\ &= \sigma_0 \frac{R}{H^2} \cdot \frac{H^3}{3} \cdot \frac{1}{2} \cdot 2\pi \\ &= \frac{\sigma_0}{3} \pi RH \end{aligned}$$

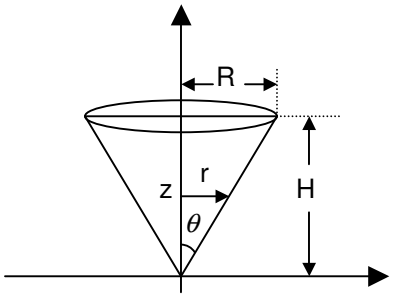
5. A surface charge density σ(ϕ, z) is distributed over the surface of a cone with height H and base radius R The cone is oriented "upside down" in a cylindrical coordinate system so that its tip is at the origin and its base is at z=H. The functional dependence of σ(ϕ, z) over the surface of the cone is:

$$\sigma(\phi, z) = \sigma_0 \left[1 - \frac{z^2}{H^2} \right] \sin^2 \phi$$

where σ_o is a known constant with units C/m^2. Find the total charge Q on the surface of the cone.

Solution:

Solution 1 : use the cylindrical coordinate (r, θ defined in the graph)



The infinitesimal area of the cone surface at the position r, as shown in the graph, is given as, in cylindrical coordinates,

$$ds = \frac{r d\phi\, dz}{\cos\theta} \qquad here\ r = \frac{R}{H} z\ for\ this\ cone\ geometry$$

$$= \frac{\sqrt{H^2+R^2}}{H} \cdot \frac{R}{H} z d\phi\, dz$$

$$Q = \int_{z=0}^{H} \int_{\phi=0}^{2\pi} \sigma(\phi, z) ds$$

$$= \int_{z=0}^{H} \int_{\phi=0}^{2\pi} \sigma(\phi, z) \frac{\sqrt{H^2+R^2}}{H} \cdot \frac{R}{H} z\, d\phi\, dz$$

$$= \frac{R\sqrt{H^2+R^2}}{H^2} \int_{z=0}^{H} \int_{\phi=0}^{2\pi} \sigma_0 [1 - \frac{z^2}{H^2}] \sin^2\phi\ z\ d\phi\, dz$$

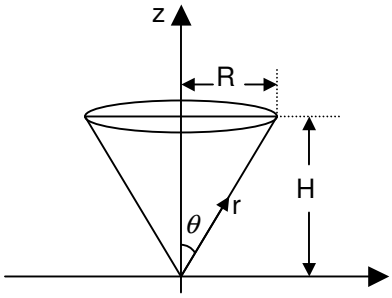
$$= \sigma_0 \frac{R\sqrt{H^2+R^2}}{H^2} \int_{z=0}^{H} [z - \frac{z^3}{H^2}] dz \int_{\phi=0}^{2\pi} \sin^2\phi d\phi$$

$$= \sigma_0 \frac{R\sqrt{H^2+R^2}}{H^2} \left[\frac{z^2}{2} - \frac{z^4}{4H^2} \right]_{z=0}^{H} \left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \right]_{\phi=0}^{2\pi}$$

$$= \sigma_0 \frac{R\sqrt{H^2+R^2}}{H^2} \left[\frac{H^2}{2} - \frac{H^2}{4} \right] \cdot \frac{1}{2} \cdot 2\pi$$

$$= \frac{\sigma_0}{4} \pi R \sqrt{H^2+R^2}$$

Solution 2: use the sphere coordinate (r, θ defined in the graph)



The infinitesimal area at the position r is given as, in spherical coordinates,

$$ds = r \sin\theta\, d\phi\, dr$$

$$= r \frac{R}{\sqrt{H^2+R^2}} d\phi\, dr$$

$$now\ z = r\cos\theta = r\frac{H}{\sqrt{H^2+R^2}}$$

$$Q = \int_{z=0}^{H} \int_{\phi=0}^{2\pi} \sigma(\phi, z) ds$$

$$= \int_{r=0}^{\sqrt{H^2+R^2}} \int_{\phi=0}^{2\pi} \sigma(\phi, r)\, r \sin\theta\ d\phi\, dz$$

$$= \int_{r=0}^{\sqrt{H^2+R^2}} \int_{\phi=0}^{2\pi} \sigma_0 [1 - \frac{r^2}{H^2} \cdot \frac{H^2}{H^2+R^2}] \sin^2\phi\ r \sin\theta\ d\phi\, dr$$

$$= \sigma_0 \frac{R}{\sqrt{H^2+R^2}} \int_{r=0}^{\sqrt{H^2+R^2}} [r - \frac{r^3}{H^2+R^2}] dr \int_{\phi=0}^{2\pi} \sin^2\phi \, d\phi$$

$$= \sigma_0 \frac{R}{\sqrt{H^2+R^2}} \left[\frac{r^2}{2} - \frac{r^4}{4\left(H^2+R^2\right)} \right]_{r=0}^{\sqrt{H^2+R^2}} \left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \right]_{\phi=0}^{2\pi}$$

$$= \sigma_0 \frac{R}{\sqrt{H^2+R^2}} \left[\frac{\left(\sqrt{H^2+R^2}\right)^2}{2} - \frac{\left(\sqrt{H^2+R^2}\right)^4}{4\left(H^2+R^2\right)} \right] \cdot \frac{1}{2} \cdot 2\pi$$

$$= \frac{\sigma_0}{4} \pi R \sqrt{H^2+R^2}$$

6. A volume charge density ρ(r, θ) is distributed within the volume of a spherical shell with the inner radius a and the outer radius b (no charge exists in the inner sphere). The functional dependence of ρ(r, θ) upon radius and angular position with respect to the vertical axis is:

$$\rho(r,\theta) = \begin{cases} 0 & r < a \\ \rho_0 \frac{b-a}{r^3} \cos^2\theta & a \le r \le b \end{cases}$$

where ρ_o is a known constant with units C/m. Find the total charge Q within the sphere.

Solution:

$$Q = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=a}^{b} \rho(r,\theta) r^2 \sin\theta \, dr d\theta \, d\phi$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=a}^{b} \rho_0 \frac{b-a}{r^3} \cos^2\theta \, r^2 \sin\theta \, dr d\theta \, d\phi$$

$$= \rho_0 (b-a) \int_{\theta=0}^{\pi} \cos^2\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \int_{r=a}^{b} \frac{1}{r} dr$$

$$= \rho_0 (b-a) \left[-\frac{\cos^3\theta}{3} \right]_{\theta=0}^{\pi} [\phi]_{\phi=0}^{2\pi} [\ln r]_{r=a}^{b}$$

$$= \rho_0 (b-a) \cdot \frac{2}{3} \cdot 2\pi \cdot \ln\frac{b}{a}$$

$$= \frac{4}{3} \pi \rho_0 (b-a) \ln\frac{b}{a}$$

7. A volume charge density ρ(r, z) is distributed within the volume of a cone with a flat base of radius R that sits at z = 0 (i.e., in the horizontal plane which cuts through the origin of the z-axis) and tapers off to a radius of zero at z = H. The functional dependence of ρ(r, z) upon position within the volume is:

$$\rho(r,z) = \rho_0 \left[1 - \left(\frac{rz}{RH} \right)^2 \right]$$

where ρ_o is a known constant with units C/m^3. Find the total charge Q within the cone.

Solution:

$$Q = \int_{z=0}^{H}\int_{\phi=0}^{2\pi}\int_{r=0}^{\frac{R}{H}(H-z)} \rho(r,z) r\, dr d\phi\, dz$$

$$= \int_{z=0}^{H}\int_{\phi=0}^{2\pi}\int_{r=0}^{\frac{R}{H}(H-z)} \rho_0\left[1-\left(\frac{rz}{RH}\right)^2\right] r\, dr d\phi\, dz$$

$$= 2\pi\,\rho_0 \int_{z=0}^{H}\int_{r=0}^{\frac{R}{H}(H-z)}\left[r-\frac{r^3z^2}{R^2H^2}\right] dr dz$$

$$= 2\pi\,\rho_0 \int_{z=0}^{H}\left\{\left[\frac{r^2}{2}-\frac{r^4z^2}{4R^2H^2}\right]_{r=0}^{\frac{R}{H}(H-z)}\right\} dz$$

$$= 2\pi\,\rho_0 \int_{z=0}^{H}\left[\frac{R^2(H-z)^2}{2H^2}-\frac{R^2(H-z)^4z^2}{4H^6}\right] dz$$

$$= \pi\,\rho_0 \frac{R^2}{H^2}\left[\int_{z=0}^{H}(H-z)^2 dz - \int_{z=0}^{H}\frac{(H-z)^4z^2}{2H^4} dz\right]$$

set (t = H-z) and substitute it into above equation.

$$Q = \pi\,\rho_0 \frac{R^2}{H^2}\left[-\int_{=H}^{0} t^2 dt + \int_{=H}^{0}\frac{t^4(H-t)^2}{2H^4} dt\right]$$

$$= \pi\,\rho_0 \frac{R^2}{H^2}\left[-\int_{=H}^{0} t^2 dt + \int_{=H}^{0}\frac{t^4H^2-2Ht^5+t^6}{2H^4} dt\right]$$

$$= \pi\,\rho_0 \frac{R^2}{H^2}\left[-\frac{t^3}{3}+\frac{1}{2H^4}\left(\frac{t^5H^2}{5}-\frac{2t^6H}{6}+\frac{t^7}{7}\right)\right]_{t=H}^{0}$$

$$= \pi\,\rho_0 \frac{R^2}{H^2}\left[\frac{H^3}{3}-\frac{1}{2H^4}\left(\frac{H^5H^2}{5}-\frac{2H^6H}{6}+\frac{H^7}{7}\right)\right]$$

$$= \frac{23}{70}\pi\,\rho_0 R^2 H$$