Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

Problem Set 7

<u>Section 6.6 p. 349-350 : Cramer's Rule and determinants.</u>

1. Problem 9

$$\det A = \begin{vmatrix} m & n & p \\ p & m & n \\ n & p & m \end{vmatrix}$$

Let us expand the determinant in terms of the first column.

$$\det A = m\begin{vmatrix} m & n \\ p & m \end{vmatrix} - p\begin{vmatrix} n & p \\ p & m \end{vmatrix} + n\begin{vmatrix} n & p \\ m & n \end{vmatrix}$$

$$= m(m^2 - pn) - p(nm - p^2) + n(n^2 - mp)$$

$$= m^3 + p^3 + n^3 - 3mpn$$

2. Problem 12 (choose your row or column wisely!)

$$\det A = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix} \qquad R_2 -> R_2 - 2R_1$$

So we can use Row 1 to eliminate the two non-zero value of the Row 2. From Theorem 1(b) (page 345), we know this won't change the value of the determinant. Now we can expand the determinant in terms of the first column.

$$\det A = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 9 & 2 \\ 0 & 2 & 16 \end{vmatrix}$$

Then expand again in terms of the first column.

$$\det A = -2\begin{vmatrix} 2 & 0 \\ 2 & 16 \end{vmatrix} = -64$$

3. Problem 18

$$x + 2y + 3z = 20$$
$$7x + 3y + z = 13$$
$$x + 6y + 2z = 0$$

Cramer's rule:

$$D_x = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} \qquad \textit{Expand by 1st column}$$
$$= 20[3\cdot 2 - 6\cdot 1] - 13[2\cdot 2 - 6\cdot 3]$$
$$= 182$$

$$D_y = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} \qquad \textit{Expand by 2nd column}$$
$$= -20[7\cdot 2 - 1\cdot 1] + 13[1\cdot 2 - 1\cdot 3]$$
$$= -273$$

$$D_z = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} \qquad \textit{Expand by 3rd column}$$
$$= 20[7\cdot 6 - 1\cdot 3] - 13[1\cdot 6 - 1\cdot 2]$$
$$= 728$$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} \qquad \textit{Expand by 1st column}$$
$$= [3\cdot 2 - 6\cdot 1] - 7[2\cdot 2 - 6\cdot 3] + [2\cdot 1 - 3\cdot 3]$$
$$= 91$$

$$\therefore x = \frac{D_x}{D} = \frac{182}{91} = 2, \quad y = \frac{D_y}{D} = \frac{-273}{91} = -3, \quad z = \frac{D_z}{D} = \frac{728}{91} = 8$$

Gauss Elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 7 & 3 & 1 & 13 \\ 1 & 6 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & -11 & -20 & -127 \\ 1 & 6 & 2 & 0 \end{array}\right) \qquad R_2 -> R_2 - 7R_1$$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & 20 \\ 0 & -11 & -20 & -127 \\ 0 & 4 & -1 & -20 \end{array}\right) \qquad R_3 -> R_3 - R_1$$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & 20 \\ 0 & -11 & -20 & -127 \\ 0 & 0 & 1 & 8 \end{array}\right) \qquad R_3 -> \frac{11R_3 + 4R_2}{-91}$$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & 20 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right) \qquad R_2 -> \frac{R_2 + 20R_3}{-11}$$

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right) \qquad R_1 -> R_1 - 2R_2 - 3R_3$$

$\therefore x = 2, \quad y = -3, \quad z = 8$

Section 6.7 p. 357-358: Matrix Inverses using Gauss-Jordan and determinants

Unlike what the book says, calculate the matrix inverse (if it exists) of the matrices in these problems by both the Gauss-Jordan and the determinant methods, and show that you get the same result.

1. Problem 1

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Gauss Jordan:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \qquad R_2 -> 2R_2 - 5R_1$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \qquad R_3 -> 2R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \qquad R_2 -> \frac{R_2 - 5R_3}{2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \qquad R_1 -> \frac{R_1 + R_3}{2}$$

Thus the inverse of the matrix is

$$\therefore A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

Determinant method:

$$\det A = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} \qquad \textit{Expand by 1st column}$$

$$= 2(1\cdot 3 - 1\cdot 0) - 5(0\cdot 3 - 1\cdot(-1))$$

$$= 1$$

Cofactors:

$$A_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = 3, \quad A_{12} = -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = -15, \quad A_{13} = \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = 5$$

$$A_{21} = -\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} = -1, \quad A_{22} = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 6, \quad A_{23} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{31} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \quad A_{32} = -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = -5, \quad A_{33} = \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = 2$$

$$\therefore A^{-1} = \frac{[A_{jk}]^T}{\det A} = \frac{\begin{pmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{pmatrix}^T}{1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

2. Problem 2

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{pmatrix}$$

Gauss Jordan Elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 4 & 11 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right) \qquad R_3 -> R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right) \qquad R_2 -> -R_2 + 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 19 & 2 & -9 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right) \qquad R_1 -> R_1 - 2R_2 - 5R_3$$

$$\therefore A^{-1} = \begin{pmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix}$$

Determinant method:

$$\det A = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{vmatrix} \qquad \mathit{Expand\ by}\ 1\mathit{st\ column}$$

$$= (-1\cdot 11 - 4\cdot 2) + 2(2\cdot 2 + 1\cdot 5)$$

$$= -1$$

Cofactors:

$$A_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 11 \end{vmatrix} = -19, \quad A_{12} = -\begin{vmatrix} 0 & 2 \\ 2 & 11 \end{vmatrix} = 4, \quad A_{13} = \begin{vmatrix} 0 & -1 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{21} = -\begin{vmatrix} 2 & 5 \\ 4 & 11 \end{vmatrix} = -2, \quad A_{22} = \begin{vmatrix} 1 & 5 \\ 2 & 11 \end{vmatrix} = 1, \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = 9, \quad A_{32} = -\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = -2, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$$

$$\therefore A^{-1} = \frac{[A_{jk}]^T}{\det A} = \frac{\begin{pmatrix} -19 & 4 & 2 \\ -2 & 1 & 0 \\ 9 & -2 & -1 \end{pmatrix}^T}{-1} = \begin{pmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix}$$

3. Problem 3

$$A = \begin{pmatrix} 3 & -1 & 5 \\ 2 & 6 & 4 \\ 5 & 5 & 9 \end{pmatrix}$$

Gauss Jordan Elimination:

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 5 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ 5 & 5 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 5 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right) \qquad R_3 -> R_3 - (R_1 + R_2)$$

We have a row of zeros in the left side matrix. So it cannot be reduced to an identity matrix by row transformations. So the inverse does not exist.

Determinant method:

$$\det A = \begin{vmatrix} 3 & -1 & 5 \\ 2 & 6 & 4 \\ 5 & 5 & 9 \end{vmatrix} \qquad \textit{Expand by } 1\textit{st column}$$

$$= 3(6\cdot 9 - 5\cdot 4) - 2(-1\cdot 9 - 5\cdot 5) + 5(-1\cdot 4 - 6\cdot 5)$$

$$= 0$$

Thus, determinant vanishes indicating that the inverse does not exist.

4. Problem 9

$$A = \begin{pmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Gauss Jordan Elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 8 & -7 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 8 & -7 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad R_2 -> R_2 - 3R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -8 & 31 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad R_1 -> R_1 - 8R_2 + 7R_3$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -8 & 31 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Determinant method:

$$\det A = \begin{vmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} \qquad \textit{Expand by } 1\textit{st column}$$

$$= 1\cdot 1 - 0\cdot 3$$

$$= 1$$

Cofactors:

$$A_{11} = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1, \quad A_{12} = -\begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} = 0, \quad A_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} 8 & -7 \\ 0 & 1 \end{vmatrix} = -8, \quad A_{22} = \begin{vmatrix} 1 & -7 \\ 0 & 1 \end{vmatrix} = 1, \quad A_{23} = -\begin{vmatrix} 1 & 8 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} 8 & -7 \\ 1 & 3 \end{vmatrix} = 31, \quad A_{32} = -\begin{vmatrix} 1 & -7 \\ 0 & 3 \end{vmatrix} = -3, \quad A_{33} = \begin{vmatrix} 1 & 8 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore A^{-1} = \frac{[A_{jk}]^T}{\det A} = \frac{\begin{pmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 31 & -3 & 1 \end{pmatrix}^T}{1} = \begin{pmatrix} 1 & -8 & 31 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Section 7.1 p. 375 : Eigenvalues and Eigenvectors.

1. **Problem 7**

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 3$$

Eigenvectors:

For λ_1:

$$(A - \lambda_1 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 1-1 & 2 \\ 0 & 3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad R_1 -> \frac{R_1}{2}, \quad R_2 -> \frac{R_2}{2}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad R_2 -> R_2 - R_1$$

$$\Rightarrow x_2 = 0$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note: There are other solutions possible. But they are all linearly dependent to the solution that we have choosen.

For λ_2:

$$(A-\lambda_2 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 1-3 & 2 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad R_1 -> \frac{R_1}{2}$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. Problem 11

$$A = \begin{pmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues:

$$\det(A-\lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)(4-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \quad \lambda_1 = 3, \quad \lambda_2 = 4, \quad \lambda_3 = 1$$

Eigenvectors:

For λ_1:

$$(A-\lambda_1 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 3-3 & 5 & 3 \\ 0 & 4-3 & 6 \\ 0 & 0 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> \frac{R_3}{-2}$$

$$\Rightarrow \begin{pmatrix} 0 & 5 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> R_2 - 6R_3$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 -> R_1 - 5R_2 - 3R_3$$

$$\Rightarrow x_2 = 0, \quad x_3 = 0$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For λ_2:

$$(A - \lambda_2 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 3-4 & 5 & 3 \\ 0 & 4-4 & 6 \\ 0 & 0 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> \frac{R_3}{-3}$$

$$\Rightarrow \begin{pmatrix} -1 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> R_2 - 6R_3$$

$$\Rightarrow \begin{pmatrix} -1 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 -> R_1 - 3R_3$$

$$\Rightarrow x_3 = 0, \quad -x_1 + 5x_2 = 0$$

$$\Rightarrow x = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

For λ_3:

$$(A - \lambda_3 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 3-1 & 5 & 3 \\ 0 & 4-1 & 6 \\ 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> \frac{R_2}{3}$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 -> R_1 - 5R_2$$

$$\Rightarrow 2x_1 - 7x_3 = 0, \quad x_2 + 2x_3 = 0$$

The system cannot be simplified further. Thus, we end up with two linearly independent equations with three unknowns. There are thus infinitely many solutions. But all the solutions are mutually linearly dependent. Choose $x_1=7$. Therefore $x_3=2$ and $x_2=-4$.

$$\Rightarrow x = \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$$

3. Problem 12

$$A = \begin{pmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{pmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0 \quad \textit{Expand by } 1\textit{st column}$$

$$\Rightarrow (a-\lambda)[(a-\lambda)(a-\lambda)-1] - (a-\lambda) = 0$$

$$\Rightarrow (a-\lambda)[(a-\lambda)^2 - 2] = 0$$

$$\Rightarrow (a-\lambda)(a-\lambda+\sqrt{2})(a-\lambda-\sqrt{2}) = 0$$

$$\Rightarrow \lambda_1 = a, \quad \lambda_2 = a+\sqrt{2}, \quad \lambda_3 = a-\sqrt{2}$$

Eigenvectors:

For λ_1:

$$(A - \lambda_1 I)x = 0$$

$$\Rightarrow \begin{pmatrix} a-a & 1 & 0 \\ 1 & a-a & 1 \\ 0 & 1 & a-a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> R_3 - R_1$$

$$\Rightarrow x_1 + x_3 = 0, \quad x_2 = 0$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For λ_2:

$$(A - \lambda_2 I)x = 0$$

$$\Rightarrow \begin{pmatrix} a-(a+\sqrt{2}) & 1 & 0 \\ 1 & a-(a+\sqrt{2}) & 1 \\ 0 & 1 & a-(a+\sqrt{2}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> \sqrt{2}R_2 + R_1$$

$$\Rightarrow \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> R_2 + R_3$$

$$\Rightarrow -\sqrt{2}x_1 + x_2 = 0, \quad -x_2 + \sqrt{2}x_3 = 0$$

$$choose\ x_1 = 1, \quad \Rightarrow \quad x_2 = \sqrt{2}, x_3 = 1$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

For λ_3:

$$(A - \lambda_3 I)x = 0$$

$$\Rightarrow \begin{pmatrix} a-(a-\sqrt{2}) & 1 & 0 \\ 1 & a-(a-\sqrt{2}) & 1 \\ 0 & 1 & a-(a-\sqrt{2}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 0 & -1 & -\sqrt{2} \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> -\sqrt{2}R_2 + R_1$$

$$\Rightarrow \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 0 & -1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> R_2 + R_3$$

$$\Rightarrow \sqrt{2}x_1 + x_2 = 0, \quad -x_2 - \sqrt{2}x_3 = 0$$

$$choose\ x_1 = 1, \quad \Rightarrow \quad x_2 = -\sqrt{2},\ x_3 = 1$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Chapter 7 review, p. 398: More Eigenvalues and Eigenvectors.

1. Problem 14

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \end{pmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & -3 & 3-\lambda \end{vmatrix} = 0 \quad Expand\ by\ 1st\ column$$

$$\Rightarrow \quad (1-\lambda)(-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 3$$

Eigenvectors:

For λ_1:

$$(A - \lambda_1 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ 0 & 0-1 & 0 \\ 0 & -3 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> R_2 - R_1$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> R_3 - 3R_1$$

$$\Rightarrow x_2 = 0, \quad x_3 = 0$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For λ_2:

$$(A - \lambda_2 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 1-0 & -1 & 0 \\ 0 & 0-0 & 0 \\ 0 & -3 & 3-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> \frac{R_3}{3}$$

$$\Rightarrow x_1 - x_2 = 0, \quad -x_2 + x_3 = 0$$

$$choose\ x_1 = 1, \quad \Rightarrow \quad x_2 = 1, x_3 = 1$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For λ_3:

$$(A - \lambda_3 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 1-3 & -1 & 0 \\ 0 & 0-3 & 0 \\ 0 & -3 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> R_3 - R_2$$

$$\Rightarrow \begin{pmatrix} -2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> \frac{R_2}{-3}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 -> \frac{R_1 + R_2}{-2}$$

$$\Rightarrow x_1 = 0, \quad x_2 = 0$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. Problem 16

$$A = \begin{pmatrix} 9 & -10 & 2 \\ -6 & 5 & 2 \\ 6 & -2 & 1 \end{pmatrix}$$

Eiganvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 9-\lambda & -10 & 2 \\ -6 & 5-\lambda & 2 \\ 6 & -2 & 1-\lambda \end{vmatrix} = 0 \quad \textit{Expand by 1st column}$$

$$\Rightarrow (9-\lambda)[(5-\lambda)(1-\lambda)+4] - (-6)[(-10)(1-\lambda)+4] + 6[-20-2(5-\lambda)] = 0$$

$$\Rightarrow (9-\lambda)[\lambda^2 - 6\lambda + 5 + 4] - 60 + 60\lambda + 24 - 120 - 60 + 12\lambda = 0$$

$$\Rightarrow (9-\lambda)(3-\lambda)^2 - 72(3-\lambda) = 0$$

$$\Rightarrow (3-\lambda)[(9-\lambda)(3-\lambda) - 72] = 0$$

$$\Rightarrow (3-\lambda)[-45 - 12\lambda + \lambda^2] = 0$$

$$\Rightarrow (3-\lambda)(15-\lambda)(-3-\lambda) = 0$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = 15, \quad \lambda_3 = -3$$

Eigenvectors:

For λ_1:

$$(A - \lambda_1 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 9-3 & -10 & 2 \\ -6 & 5-3 & 2 \\ 6 & -2 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & -10 & 2 \\ -6 & 2 & 2 \\ 6 & -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & -10 & 2 \\ -6 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> R_3 + R_2$$

$$\Rightarrow \begin{pmatrix} 6 & -10 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> \frac{R_1 + R_2}{4}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 -> \frac{R_1 - 2R_2}{6}$$

$$\Rightarrow x_1 - x_2 = 0, \quad -2x_2 + x_3 = 0$$

$$choose\ x_1 = 1, \quad \Rightarrow \quad x_2 = 1, x_3 = 2$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

For λ_2:

$$(A - \lambda_2 I)x = 0$$

$$\Rightarrow \begin{pmatrix} 9-15 & -10 & 2 \\ -6 & 5-15 & 2 \\ 6 & -2 & 1-15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -6 & -10 & 2 \\ -6 & -10 & 2 \\ 6 & -2 & -14 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -6 & -10 & 2 \\ 0 & 0 & 0 \\ 6 & -2 & -14 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 -> -R_1 + R_2$$

$$\Rightarrow \begin{pmatrix} -6 & -10 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_3 -> \frac{R_1 + R_3}{-12}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad R_1 -> \frac{R_1 - 2R_3}{-6}$$

$$\Rightarrow x_1 + 2x_2 = 0, \quad x_2 + x_3 = 0$$

$$choose\ x_1 = 2, \quad \Rightarrow \quad x_2 = -1, x_3 = 1$$

$$\Rightarrow x=\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

For λ_3:

$$(A-\lambda_3 I)x=0$$

$$\Rightarrow\begin{pmatrix}9-(-3) & -10 & 2\\ -6 & 5-(-3) & 2\\ 6 & -2 & 1-(-3)\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}12 & -10 & 2\\ -6 & 8 & 2\\ 6 & -2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}12 & -10 & 2\\ 0 & 0 & 0\\ 6 & -2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad R_2->R_2+R_1-R_3$$

$$\Rightarrow\begin{pmatrix}12 & -10 & 2\\ 0 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad R_3->\frac{-R_1+2R_3}{6}$$

$$\Rightarrow\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad R_1->\frac{R_1-2R_3}{12}$$

$$\Rightarrow x_1-x_2=0,\quad x_2+x_3=0$$

$$choose\ x_1=1,\quad\Rightarrow\quad x_2=1,\ x_3=-1$$

$$\Rightarrow x=\begin{pmatrix}1\\1\\-1\end{pmatrix}$$