# Carnegie Mellon University

## Department of Electrical and Computer Engineering

**18-200** **Fall 2000**

### Mathematical Foundations of Electrical Engineering

### Problem Set 5 – Solution

1. For this problem, ( sinusoidal driving function),
   y’’ + 4y’ + 4y = -5 sin (3t) + 12 cos (3t); y(0) = 1; y’(0) = 3

***particular solution***

a) the phasor method

The driving function is a sum of harmonic functions of angular frequency 3. So it can be represented in the form $Re[(P+jQ)e^{3jt}]$. Here $P$ and $Q$ are real numbers.

$$\begin{aligned}-5\sin(3t)+12\cos(3t) &= \mathrm{Re}[(P+jQ)e^{3jt}]\\ &= \mathrm{Re}[(P+jQ)(\cos(3t)+j\sin(3t))]\\ &= P\cos(3t)-Q\sin(3t)\end{aligned}$$

$$\Rightarrow \quad P=12, \quad Q=5$$

Therefore, the differential equation can be re-written as

$$y''+4y'+4y=\mathrm{Re}[(12+j5)e^{3jt}]$$

The form of the equation suggests that the particular solution is also a harmonic function of angular frequency 3. So we can represent the particular solution in the form $y_p(t) = Re[Y_0e^{3jt}]$. Here $Y_0$ is a complex constant. Therefore

$$y'_p=\mathrm{Re}[3jY_0e^{3jt}], \quad y''_p=\mathrm{Re}[-9Y_0e^{3jt}]$$

Substituting these into the differential equation,

$$\mathrm{Re}[-9Y_0e^{3jt}]+4\,\mathrm{Re}[3jY_0e^{3jt}]+4\,\mathrm{Re}[Y_0e^{3jt}]=\mathrm{Re}[(12+j5)e^{3jt}]$$

$$\Rightarrow \quad \mathrm{Re}[Y_0(-5+12j)e^{3jt}]=\mathrm{Re}[(12+j5)e^{3jt}]$$

We realize that any solution to the following equation is also a solution for the previous equation.

$$Y_0(-5+12j)e^{3jt}=(12+j5)e^{3jt}$$

$$Y_0=\frac{12+j5}{-5+12j}=-j\frac{(12j-5)}{-5+12j}=-j$$

$$\Rightarrow y_p=\mathrm{Re}[Y_0e^{3jt}]=\mathrm{Re}[-je^{3jt}]=\sin(3t)$$

b) the method of undetermined coefficients (assume a solution of the form Kcos(3t) + Msin(3t)).

The driving function indicates that a good choice for the particular solution is a trigonometric expression. Let

$$y_p=K\cos(3t)+M\sin(3t)$$

Therefore,

$$y'_p=-3K\sin(3t)+3M\cos(3t)$$

$$y''_p=-9K\cos(3t)-9M\sin(3t)$$

Substituting into the equation

$$[-9K\cos(3t)-9M\sin(3t)]+4[-3K\sin(3t)+3M\cos(3t)]$$
$$+4[K\cos(3t)+M\sin(3t)]=-5\sin(3t)+12\cos(3t)$$

$$\Rightarrow \quad \begin{cases}-5K+12M=12 \\ -12K-5M=-5\end{cases} \quad \Rightarrow \quad \begin{cases}K=0 \\ M=1\end{cases}$$

$$\Rightarrow \quad y_p=\sin(3t)$$

Thus, by both the methods, we get the same particular solution.

***Homogeneous Solution***

c) Also solve for the homogeneous solution and find the total solution for y(t).

This is the solution of the differential equation obtained by setting the driving term zero.

$$y_h''+4y_h'+4y_h=0$$

The characteristic equation is then,

$$\lambda^2+4\lambda+4=0$$

$$\Rightarrow \quad \lambda_1=\lambda_2=-2$$

The characteristic equation has two real roots of the same value –2. Therefore, the solution has the form,

$$y_h(t)=c_1e^{-2t}+c_2te^{-2t}$$

and thus the total solution is,

$$y(t)=y_h(t)+y_p(t)=c_1e^{-2t}+c_2te^{-2t}+\sin(3t)$$

and,

$$y'(t)=-2c_1e^{-2t}+c_2e^{-2t}-2c_2te^{-2t}+3\cos(3t)=(-2c_1+c_2-2c_2t)e^{-2t}+3\cos(3t)$$

Initial values give,

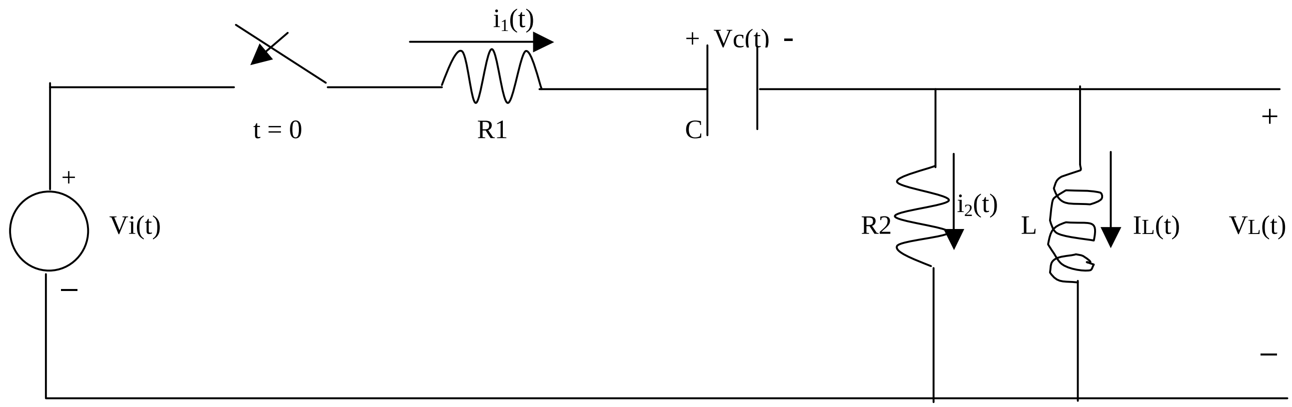
$$\begin{cases}y(0)=c_1=1 \\ y'(0)=-2c_1+c_2+3=3\end{cases} \quad \Rightarrow \quad \begin{cases}c_1=1 \\ c_2=2\end{cases}$$

Therefore

$$y(t)=e^{-2t}+2te^{-2t}+\sin(3t)$$

2. This next problem is an L-R-C circuit with a sinusoidal driving function (voltage source) attached.

For this problem, we are going to solve for the voltage in the capacitor, Vc(t), and current in the inductor, IL(t). The switch is closed at time t=0, and the initial conditions on Vc(t) and IL(t) are: Vc(0) = 0v; IL(0) = 0A.




The circuit diagram is shown in the figure above. The diagram indicates the potentials and the currents.

a) Write two second-order non-homogeneous ordinary differential equations (ODE) for the circuit above, one for Vc(t) and one for IL(t). Your answers should be in terms of R1, R2, L, C, Vi(t) and Vi'(t) for both equations, and either Vc(t) (and its derivatives), or IL(t) (and its derivatives) depending on the quantity for which you are writing the ODE.

i) Use Ohm's law to write an equation for the current through R1 (also the current through C) in terms of R1 and voltages Vi(t), Vc(t), and VL(t).

$$i_1(t)R_1 = V_i(t) - V_C(t) - V_L(t)$$

ii) Write an equation for the current through the capacitor (C) in terms of C and Vc(t)

$$i_1(t) = CV_C'(t)$$

iii) Write an equation for the voltage across the inductor (L) in terms of L and the current through it.

$$V_L(t) = L\,I_L'(t)$$

iv) Write a KCL equation at the node where C, L, and R2 come together.

$$i_1(t) - i_2(t) - I_L(t) = 0$$

v) Use Ohm's law to write an equation for the current through R2 in terms of VL(t) and R2.

$$i_2(t)R_2 = V_L(t)$$

Do some equation combining to eliminate $V_L(t)$, $i_1(t)$, $i_2(t)$, we can get,
for $V_C(t)$:

$$L\left(1+\frac{R_1}{R_2}\right)V_C''(t) + \left(R_1 + \frac{L}{R_2C}\right)V_C'(t) + \frac{1}{C}V_C(t) = \frac{L}{R_2C}V_i'(t) + \frac{1}{C}V_i(t)$$

And for $I_L(t)$:

$$L\left(1+\frac{R_1}{R_2}\right)I_L''(t) + \left(R_1 + \frac{L}{R_2C}\right)I_L'(t) + \frac{1}{C}I_L(t) = V_i'(t)$$

b) Solve the characteristic equation for the above ODE and express your λ's in terms of the circuit components: L, C, R1, and R2.

The characteristic equation for the homogeneous part:

$$L\left(1+\frac{R_1}{R_2}\right)\lambda^2+\left(R_1+\frac{L}{R_2C}\right)\lambda+\frac{1}{C}=0$$

$$\Rightarrow \lambda=\frac{-\left(R_1+\frac{L}{R_2C}\right)\pm\sqrt{\left(R_1+\frac{L}{R_2C}\right)^2-4L\left(1+\frac{R_1}{R_2}\right)\frac{1}{C}}}{2L\left(1+\frac{R_1}{R_2}\right)}$$

c) As in lecture, let L = 100 microhenrys and C = 100 picofarads. If R2 = infinity (open circuit, or removing R2 from the circuit), what range(s) of R1 make the system “over damped” and “under damped”? What is the value of R1 you need to make the system “critically damped”?

The discriminant (the expression under the square root sign) of the quadratic equation ($\Delta$) is given by,

$$\Delta=\left(R_1+\frac{L}{R_2C}\right)^2-4L\left(1+\frac{R_1}{R_2}\right)\frac{1}{C}$$

Since $R_2$ is infinite, this reduces to

$$\begin{aligned}\Delta&=R_1^2-\frac{4L}{C}\\&=R_1^2-\frac{4\times100\times10^{-6}H}{100\times10^{-12}F}\\&=R_1^2-4\times10^6\end{aligned}$$

Thus,

$R_1 > 2k\Omega$: Over-damped case (two different real roots).
$R_1 < 2k\Omega$: Under-damped case (complex conjugate roots).
$R_1 = 2k\Omega$: Critically damped case (two identical real roots).

d) If the values of L and C are as in part c) and R1 = 0 ohms (short circuit), what range(s) of R2 make the system “over damped” and “under damped”? What is the value of R2 you need to make the system “critically damped”?

Now the discriminant is reduced to

$$\begin{aligned}\Delta&=\left(\frac{L}{R_2C}\right)^2-\frac{4L}{C}\\&=\left(\frac{100\times10^{-6}}{R_2\cdot100\times10^{-12}}\right)^2-\frac{4\times100\times10^{-6}}{100\times10^{-12}}\\&=\frac{10^{12}}{R_2^2}-4\times10^6\end{aligned}$$

Thus,

$R_2 < 500\Omega$: Over-damped case (two different real roots).
$R_2 > 500\Omega$: Under-damped case (complex conjugate roots).
$R_2 = 500\Omega$: Critically damped case (two identical real roots).

e) Let the values of L and C be as in part c), let R1 = 0 ohms, and let R2 = 2000 ohms. Let Vi(t) = 10v. Solve the non-homogeneous ODE’s with these values and the initial conditions given in the problem statement to find final function’s for the voltage Vc(t) and current IL(t). (You may use MATLAB or your calculator to help you with the math, but set up the equations so we can see what you’ve done.) Use MATLAB to plot the voltage Vc(t) and current IL(t) in the range of t = 0 to 10 **micro**-seconds. (use a fairly small step size so you can see what is going on, like 0.05 microseconds or less).

Let us substitute the values to find the coefficients of the differential equations.

$$L\left(1+\frac{R_1}{R_2}\right)=L=100\times10^{-6}=10^{-4}$$

$$\left(R_1+\frac{L}{R_2C}\right)=\frac{L}{R_2C}=\frac{100\times10^{-6}}{2000\times100\times10^{-12}}=500$$

$$\frac{1}{C}=\frac{1}{100\times10^{-12}}=10^{10}$$

Now, re-writing the differential equation for Vc(t) to,

$$10^{-4}V_C''(t)+500V_C'(t)+10^{10}V_C(t)=10^{10}\times10$$

and the differential equation for $I_L$(t) to,

$$10^{-4}I_L''(t)+500I_L'(t)+10^{10}I_L(t)=0$$

with the initial conditions Vc(0)=0V and $I_L$(0)=0A. Also, based on the circuit, these initial conditions imply that

$$V_L(0)=V_i(0)=10V$$

$$\therefore\frac{V_L(0)}{R_2}=i_2(0)=i_1(0)=CV_C'(0)\quad\Rightarrow\quad V_C'(0)=\frac{V_L(0)}{R_2C}=\frac{10}{2000\times100\times10^{-12}}=5\times10^7\ (V/s)$$

$$LI_L'(0)=V_L(0)\quad\Rightarrow\quad I_L'(0)=\frac{V_L(0)}{L}=\frac{10}{100\times10^{-6}}=10^5\ (A/s)$$

First, let us solve the homogeneous equation, which is the same for both Vc(t) and $I_L$(t) and given as,

$$10^{-4}\lambda^2+500\lambda+10^{10}=0$$

$$\lambda=\frac{-500\pm\sqrt{500^2-4\times10^{-4}\times10^{10}}}{2\times10^{-4}}=-2.5\times10^6\pm9.68\times10^6\,j$$

Note: In out final answer, λ always occurs along with t, i.e., the terms with λ are always of the form λt. So we can drop a factor of $10^6$ from λ to essentially rescale the time unit to μs from s. This scaling has to be done in the initial conditions also. So $V_C'(0)$ becomes 50V/μs and $I_L'(0)$ 0.1A/μs. So, with the understanding that our time unit is μs now, we have,

$$\lambda=-2.5\pm9.68\,j$$

$$y_h(t)=e^{-2.5t}[A\cos(9.68t)+B\sin(9.68t)]$$

Now let us solve the particular solution. Since the driving function for $I_L(t)$ is zero, we only need to find the particular solution for $V_C(t)$.

The driving function suggests that the particular solution is a constant. The derivatives of a constant vanish. Therefore,

$$10^{10}y_p(t)=10^{11}\quad\Rightarrow\quad y_p(t)=10$$

Thus we have,

$$V_C(t)=e^{-2.5t}[A_1\cos(9.68t)+B_1\sin(9.68t)]+10$$

$$I_L(t)=e^{-2.5t}[A_2\cos(9.68t)+B_2\sin(9.68t)]$$

and

$$V_C'(t)=-2.5e^{-2.5t}[A_1\cos(9.68t)+B_1\sin(9.68t)]+e^{-2.5t}[-9.68A_1\sin(9.68t)+9.68B_1\cos(9.68t)]$$

$$I_L'(t)=-2.5e^{-2.5t}[A_2\cos(9.68t)+B_2\sin(9.68t)]+e^{-2.5t}[-9.68A_2\sin(9.68t)+9.68B_2\cos(9.68t)]$$

Initial values give,

$$\begin{cases}V_C(0)=A_1+10=0\\ V_C'(0)=-2.5A_1+9.68B_1=50\end{cases}\quad\Rightarrow\quad\begin{cases}A_1=-10\\ B_1=2.583\end{cases}$$

$$\begin{cases} I_L(0) = A_2 = 0 \\ I'_L(0) = -2.5A_2 + 9.68B_2 = 0.1 \end{cases} \Rightarrow \begin{cases} A_2 = 0 \\ B_2 = 0.01033 \end{cases}$$

Therefore, the final solutions are,

$$V_C(t) = e^{-2.5t}[-10\cos(9.68t) + 2.583\sin(9.68t)] + 10$$

$$I_L(t) = 0.01033e^{-2.5t}\sin(9.68t)$$

(t in μs)

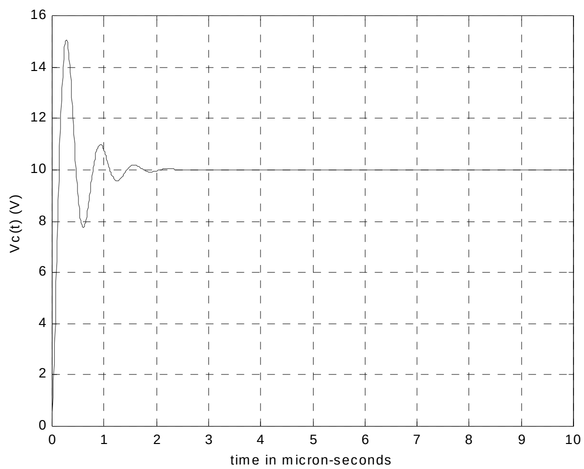
or,

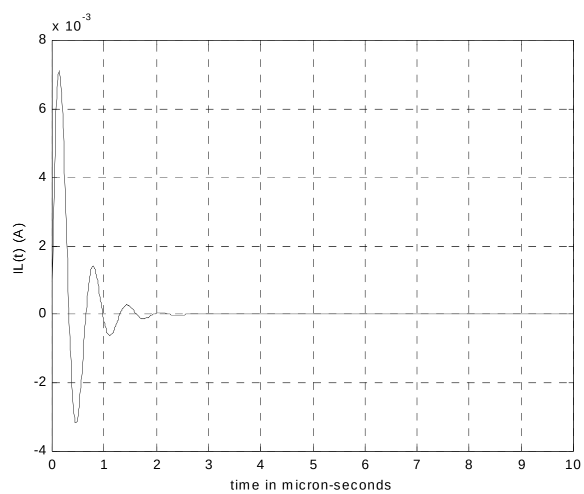
$$V_C(t) = e^{-2.5\times10^6 t}[-10\cos(9.68\times10^6 t) + 2.583\sin(9.68\times10^6 t)] + 10$$

$$I_L(t) = 0.01033e^{-2.5\times10^6 t}\sin(9.68\times10^6 t)$$

(t in seconds)

Plot $V_C(t)$ and $I_L(t)$ with matlab:




Note: This is an under-damped system. When the transients die down, the currents and voltages attain steady state. For this circuit, at the steady state, all the potential drops across the capacitor and all the currents goes to zero. Therefore, the voltage in the plot converges to the source voltage and the current converges to zero.