Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundation of Electrical Engineering

Problem Set 4 – Solution

Section 2.9, p.108

1. Problem 15 (plot y(x) between 0 and 2)

$$y'' + 1.5y' - y = 12x^2 + 6x^3 - x^4, \quad y(0) = 4, \quad y'(0) = -8$$

Homogeneous solution:

The homogeneous equation is given as

$$y_h''(x) + 1.5y_h'(x) - y_h(x) = 0$$

The characteristic equation is then:

$$\lambda^2 + 1.5\lambda - 1 = 0$$

$$\Rightarrow \quad (\lambda_1 - 0.5)(\lambda_2 + 2) = 0 \quad \Rightarrow \lambda_1 = 0.5, \lambda_2 = -2$$

The homogeneous solution is thus

$$y_h(x) = c_1 e^{0.5x} + c_2 e^{-2x}$$

Particular solution:

The right side function indicates that a good choice for the particular solution is a polynomial of the same degree as the right side function (Basic Rule A).

Let

$$y_p(x) = k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0$$

Therefore,

$$y_p'(x) = 4k_4 x^3 + 3k_3 x^2 + 2k_2 x + k_1$$

$$y_p''(x) = 12k_4 x^2 + 6k_3 x + 2k_2$$

Substitute into the equation,

$$\begin{aligned} y_p'' + 1.5y_p' - y_p &= (12k_4 x^2 + 6k_3 x + 2k_2) + 1.5(4k_4 x^3 + 3k_3 x^2 + 2k_2 x + k_1) \\ &\quad - (k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0) \\ &= (2k_2 + 1.5k_1 - k_0) + (6k_3 + 3k_2 - k_1)x + (12k_4 + 4.5k_3 - k_2)x^2 \\ &\quad + (6k_4 - k_3)x^3 - k_4 x^4 \\ &= 12x^2 + 6x^3 - x^4 \end{aligned}$$

Equating the coefficients of the same powers of *x* on both sides,

$$\begin{cases} 2k_2 + 1.5k_1 - k_0 = 0 \\ 6k_3 + 3k_2 - k_1 = 0 \\ 12k_4 + 4.5k_3 - k_2 = 12 \\ 6k_4 - k_3 = 6 \\ -k_4 = -1 \end{cases} \quad \Rightarrow \quad \begin{cases} k_0 = 0 \\ k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \\ k_4 = 1 \end{cases}$$

Therefore

$$y_p(x) = x^4$$

Combine homogeneous and particular solutions to get the general solution,

$$y(x) = y_p(x) + y_h(x) = x^4 + c_1 e^{0.5x} + c_2 e^{-2x}$$

and

$$y'(x) = 4x^3 + 0.5c_1 e^{0.5x} - 2c_2 e^{-2x}$$

Initial values give

$$\begin{cases} y(0) = c_1 + c_2 = 4 \\ y'(0) = 0.5c_1 - 2c_2 = -8 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 0 \\ c_2 = 4 \end{cases}$$

Therefore

$$y(x) = x^4 + 4e^{-2x}$$

Check:

$$y'(x) = 4x^3 - 8e^{-2x}$$

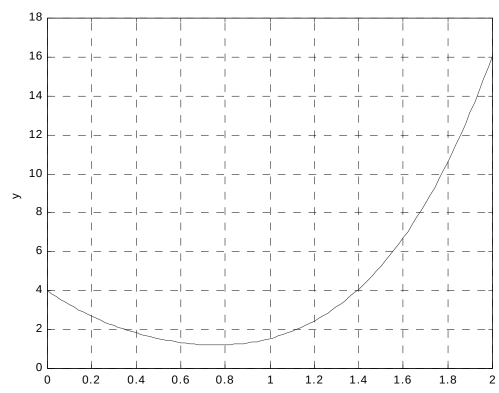
$$y''(x) = 12x^2 + 16e^{-2x}$$

$$\begin{aligned} y'' + 1.5y' - y &= (12x^2 + 16e^{-2x}) + 1.5(4x^3 - 8e^{-2x}) - (x^4 + 4e^{-2x}) \\ &= (12x^2 + 6x^3 - x^4) + (16 - 12 - 4)e^{-2x} \\ &= 12x^2 + 6x^3 - x^4 \end{aligned}$$

$$y(0) = 0^4 + 4e^0 = 4$$

$$y'(0) = 0 - 8e^0 = -8$$

Plot with matlab,



2. problem 16 (plot y(x) between 0 and 2)

$$y'' - 6y' + 13y = 4e^{3x}, \quad y(0) = 2, \quad y'(0) = 4$$

Homogeneous solution:

The homogeneous equation is given as

$$y_h''(x) - 6y_h'(x) + 13y_h(x) = 0$$

The characteristic equation is then:

$$\lambda^2 - 6\lambda + 13 = 0$$

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4\times1\times13}}{2} = 3 \pm 2i$$

$$\therefore -\frac{a}{2} = 3, \omega = 2$$

The homogeneous solution is thus

$$y_h(x) = e^{3x}(A\cos 2x + B\sin 2x)$$

Particular solution:

The right side function indicates that a good choice for the particular solution is (Basic Rule A)

$$y_p(x) = Ce^{3x}$$

Therefore,

$$y_p'(x) = 3Ce^{3x}$$

$$y_p''(x) = 9Ce^{3x}$$

Substitute into the equation,

$$y_p'' - 6y_p' + 13y_p = 9Ce^{3x} - 6\cdot 3Ce^{3x} + 13Ce^{3x} = 4Ce^{3x} = 4e^{3x}$$

$$\Rightarrow \quad C = 1$$

$$\Rightarrow \quad y_p(x) = e^{3x}$$

Combine homogeneous and particular solutions to get the general solution,

$$y(x) = y_p(x) + y_h(x) = e^{3x} + e^{3x}(A\cos 2x + B\sin 2x)$$

and

$$y'(x) = 3e^{3x} + 3e^{3x}(A\cos 2x + B\sin 2x) + e^{3x}(-2A\sin 2x + 2B\cos 2x)$$

Initial values give

$$\begin{cases} y(0) = 1 + A = 2 \\ y'(0) = 3 + 3A + 2B = 4 \end{cases} \quad \Rightarrow \quad \begin{cases} A = 1 \\ B = -1 \end{cases}$$

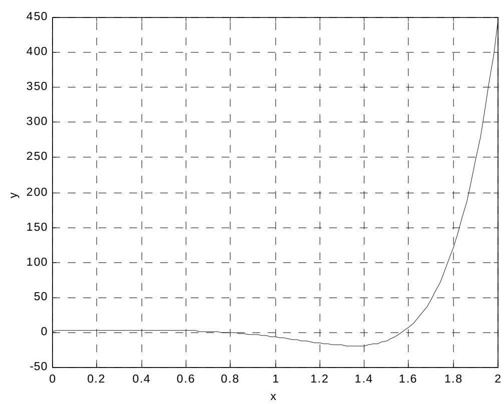
Therefore

$$y(x) = e^{3x} + e^{3x}(\cos 2x - \sin 2x) = e^{3x}(1 + \cos 2x - \sin 2x)$$

Check:

$$y'(x) = 3e^{3x}(1+\cos 2x - \sin 2x) + e^{3x}(-2\sin 2x - 2\cos 2x)$$
$$= e^{3x}(3+\cos 2x - 5\sin 2x)$$
$$y''(x) = 3e^{3x}(3+\cos 2x - 5\sin 2x) + e^{3x}(-2\sin 2x - 10\cos 2x)$$
$$= e^{3x}(9 - 7\cos 2x - 17\sin 2x)$$
$$y'' - 6y' + 13y = e^{3x}(9-7\cos 2x - 17\sin 2x) - 6e^{3x}(3+\cos 2x - 5\sin 2x)$$
$$+13e^{3x}(1+\cos 2x - \sin 2x)$$
$$= e^{3x}\left[(9-18+13) + (-7-6+13)\cos 2x + (-17+30-13)\sin 2x\right]$$
$$= 4e^{3x}$$
$$y(0) = e^0(1+1-0) = 2$$
$$y'(0) = e^0(3+1-0) = 4$$

Plot with matlab,



<u>Chapter 2 review, p.142</u>

3. Problem 31 (plot y(x) between 0 and π)

$$y'' + 16y = 17e^x, \quad y(0) = 6, \quad y'(0) = -2$$

Homogeneous solution:

The homogeneous equation is given as

$$y_h''(x) + 16y_h(x) = 0$$

The characteristic equation is then:

$$\lambda^2 + 16 = 0$$

$$\Rightarrow \lambda_1 = 4i, \lambda_2 = -4i \qquad (\therefore -\frac{a}{2} = 0, \omega = 4)$$

The homogeneous solution is thus

$$y_h(x) = A\cos 4x + B\sin 4x$$

Particular solution:

The right side function indicates that a good choice for the particular solution is (Basic Rule A)

$$y_p(x) = Ce^x$$

Therefore,

$$y_p'(x) = Ce^x$$

$$y_p''(x) = Ce^x$$

Substitute into the equation,

$$y_p'' + 16y_p = Ce^x + 16Ce^x = 17Ce^x = 17e^x$$

$$\Rightarrow \quad C = 1$$

$$\Rightarrow \quad y_p(x) = e^x$$

Combine homogeneous and particular solutions to get the general solution,

$$y(x) = y_p(x) + y_h(x) = e^x + A\cos 4x + B\sin 4x$$

and

$$y'(x) = e^x - 4A\sin 4x + 4B\cos 4x$$

Initial values give

$$\begin{cases} y(0) = 1 + A = 6 \\ y'(0) = 1 + 4B = -2 \end{cases} \quad \Rightarrow \quad \begin{cases} A = 5 \\ B = -0.75 \end{cases}$$

Therefore

$$y(x) = e^x + 5\cos 4x - 0.75\sin 4x$$

Check:

$$y'(x) = e^x - 20\sin 4x - 3\cos 4x$$

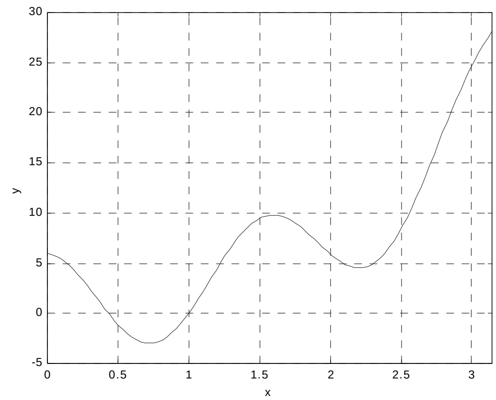
$$y''(x) = e^x - 80\cos 4x + 12\sin 4x$$

$$\begin{aligned} y'' + 16y &= (e^x - 80\cos 4x + 12\sin 4x) + 16(e^x + 5\cos 4x - 0.75\sin 4x) \\ &= 17e^x + (-80 + 80)\cos 4x + (12 - 12)\sin 4x \\ &= 17e^x \end{aligned}$$

$$y(0) = e^0 + 5 = 6$$

$$y'(0) = e^0 - 3 = -2$$

Plot with matlab,



4. Problem 32 (plot y(x) between 0 and 2π)

$$y'' - 3y' + 2y = 10\sin x, \quad y(0) = 1, \quad y'(0) = -6$$

Homogeneous solution:

The homogeneous equation is given as

$$y_h''(x) - 3y_h'(x) + 2y_h(x) = 0$$

The characteristic equation is then:

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0 \quad \Rightarrow \lambda_1 = 1, \lambda_2 = 2$$

The homogeneous solution is thus

$$y_h(x) = c_1 e^x + c_2 e^{2x}$$

Particular solution:

The right side function indicates that a good choice for the particular solution is (Basic Rule A)

$$y_p(x) = K\cos x + M\sin x$$

Therefore,

$$y_p'(x) = -K\sin x + M\cos x$$

$$y_p''(x) = -K\cos x - M\sin x$$

Substitute into the equation,

$$\begin{aligned} y_p'' - 3y_p' + 2y_p &= (-K\cos x - M\sin x) - 3(-K\sin x + M\cos x) \\ &\quad + 2(K\cos x + M\sin x) \\ &= (-K - 3M + 2K)\cos x + (-M + 3K + 2M)\sin x \\ &= 10\sin x \end{aligned}$$

Equating the coefficients of the same powers of x on both sides,

$$\begin{cases} K - 3M = 0 \\ 3K + M = 10 \end{cases} \quad \Rightarrow \quad \begin{cases} K = 3 \\ M = 1 \end{cases}$$

Therefore

$$y_p(x) = 3\cos x + \sin x$$

Combine homogeneous and particular solutions to get the general solution,

$$y(x) = y_p(x) + y_h(x) = 3\cos x + \sin x + c_1 e^x + c_2 e^{2x}$$

and

$$y'(x) = -3\sin x + \cos x + c_1 e^x + 2c_2 e^{2x}$$

Initial values give

$$\begin{cases} y(0) = 3 + c_1 + c_2 = 1 \\ y'(0) = 1 + c_1 + 2c_2 = -6 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 3 \\ c_2 = -5 \end{cases}$$

Therefore

$$y(x) = 3\cos x + \sin x + 3e^x - 5e^{2x}$$

Check:

$$y'(x) = -3\sin x + \cos x + 3e^x - 10e^{2x}$$

$$y''(x) = -3\cos x - \sin x + 3e^x - 20e^{2x}$$

$$\begin{aligned} y'' - 3y' + 2y &= (-3\cos x - \sin x + 3e^x - 20e^{2x}) - 3(-3\sin x + \cos x + 3e^x - 10e^{2x}) \\ &\quad + 2(3\cos x + \sin x + 3e^x - 5e^{2x}) \\ &= (-3 - 3 + 6)\cos x + (-1 + 9 + 2)\sin x \\ &\quad + (3 - 9 + 6)e^x + (-20 + 30 - 10)e^{2x} \\ &= 10\sin x \end{aligned}$$

$$y(0) = 3 + 3 - 5 = 1$$

$$y'(0) = 1 + 3 - 10 = -6$$

Plot with matlab,