Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

Problem Set 3 – Solution

<u>Section 2.2 p. 75</u>

1. Problem 11 (plot y(x) between 0 and 2)

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Characteristic equation:

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda_1 + 2)(\lambda_2 + 2) = 0 \quad \Rightarrow \lambda_1 = -2, \quad \lambda_2 = -2$$

The characteristic equation has two identical real roots.

$$\Rightarrow \quad y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$and \;\; y'(x) = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x} = (c_2 - 2c_1)e^{-2x} - 2c_2 x e^{-2x}$$

Initial values give:

$$\begin{cases} y(0) = c_1 = 1 \\ y'(0) = c_2 - 2c_1 = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 1 \\ c_2 = 3 \end{cases}$$

Therefore,

$$y(x) = e^{-2x} + 3xe^{-2x}$$

Check:

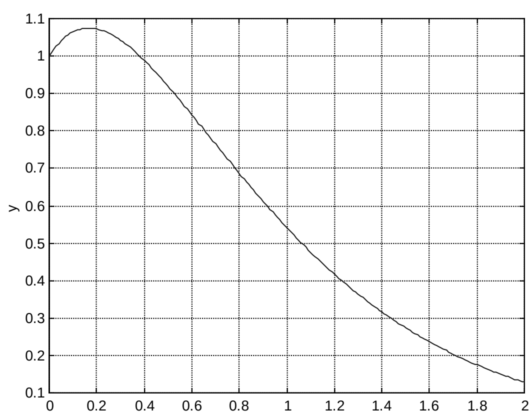
$$y'(x) = (3-2)e^{-2x} - 2 \cdot 3xe^{-2x} = e^{-2x} - 6xe^{-2x}$$

$$y''(x) = -2e^{-2x} - 6e^{-2x} + 12xe^{-2x} = -8e^{-2x} + 12xe^{-2x}$$

$$\begin{aligned} \therefore\; y'' + 4y' + 4y &= \left(-8e^{-2x} + 12xe^{-2x}\right) + 4\left(e^{-2x} - 6xe^{-2x}\right) + 4\left(e^{-2x} + 3xe^{-2x}\right) \\ &= (-8+4+4)e^{-2x} + (12-24+12)xe^{-2x} \\ &= 0 \end{aligned}$$

$$and \;\; y(0) = 1, \quad y'(0) = 1$$

Plot with Matlab:



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2. Problem 12 (plot y(x) between 0 and 2)

$$y'' - y = 0, \quad y(0) = 3, \quad y'(0) = -3$$

Characteristic equation:

$$\lambda^2 - 1 = 0$$

$$(\lambda_1 + 1)(\lambda_2 - 1) = 0 \quad \Rightarrow \lambda_1 = 1, \quad \lambda_2 = -1$$

The characteristic equation has two different real roots.

$$\Rightarrow \quad y(x) = c_1 e^{x} + c_2 e^{-x}$$

$$and \;\; y'(x) = c_1 e^{x} - c_2 e^{-x}$$

Initial values give:

$$\begin{cases} y(0) = c_1 + c_2 = 3 \\ y'(0) = c_1 - c_2 = -3 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 0 \\ c_2 = 3 \end{cases}$$

Therefore,

$$y(x) = 3e^{-x}$$

Check:

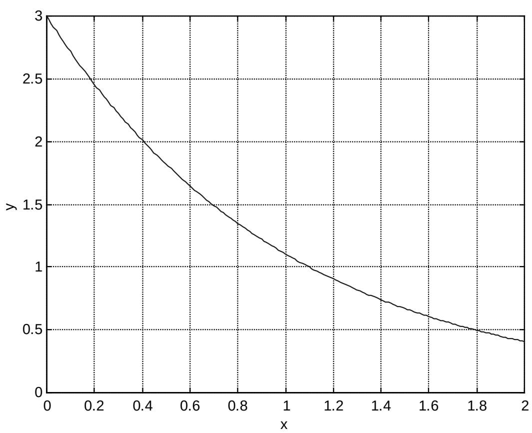
$$y'(x) = -3e^{-x}$$

$$y''(x) = 3e^{-x}$$

$$\therefore y'' - y = 3e^{-x} - 3e^{-x} = 0$$

$$and \;\; y(0) = 3, \quad y'(0) = -3$$

Plot with Matlab:



3. Problem 13 (plot y(x) between 0 and 2)

$$8y'' - 2y' - y = 0, \quad y(0) = -0.2, \quad y'(0) = -0.325$$

Characteristic equation:

$$8\lambda^2 - 2\lambda - 1 = 0$$

$$(4\lambda_1 + 1)(2\lambda_2 - 1) = 0 \quad \Rightarrow \lambda_1 = -0.25, \quad \lambda_2 = 0.5$$

The characteristic equation has two different real roots.

$$\Rightarrow \quad y(x) = c_1 e^{-0.25x} + c_2 e^{0.5x}$$

$$and \;\; y'(x) = -0.25c_1 e^{-0.25x} + 0.5c_2 e^{0.5x}$$

Initial values give:

$$\begin{cases} y(0) = c_1 + c_2 = -0.2 \\ y'(0) = -0.25c_1 + 0.5c_2 = -0.325 \end{cases} \Rightarrow \begin{cases} c_1 = 0.3 \\ c_2 = -0.5 \end{cases}$$

Therefore,

$$y(x) = 0.3e^{-0.25x} - 0.5e^{0.5x}$$

Check:

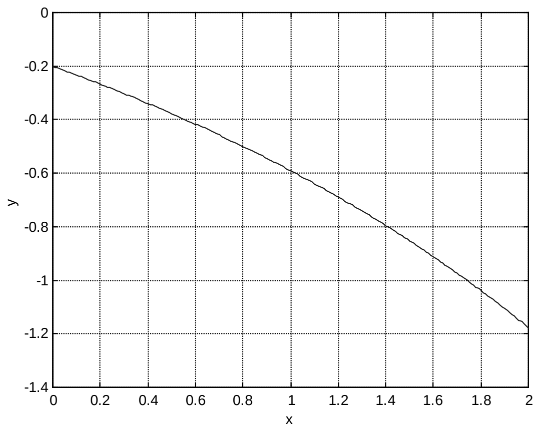
$$y'(x) = -0.075e^{-0.25x} - 0.25e^{0.5x}$$

$$y''(x) = 0.01875e^{-0.25x} - 0.125e^{0.5x}$$

$$\begin{aligned} \therefore 8y'' - 2y' - y &= 8(0.01875e^{-0.25x} - 0.125e^{0.5x}) - 2(-0.075e^{-0.25x} - 0.25e^{0.5x}) - (0.3e^{-0.25x} - 0.5e^{0.5x}) \\ &= (0.15 + 0.15 - 0.3)e^{-0.25x} + (-1 + 0.5 + 0.5)e^{0.5x} \\ &= 0 \end{aligned}$$

$$and\ \ y(0) = 0.3 - 0.5 = -0.2, \quad y'(0) = -0.075 - 0.25 = -0.325$$

Plot with Matlab:



4. Problem 14 (plot y(x) between 0 and 2)

$$4y'' - 25y = 0, \quad y(0) = 0, \quad y'(0) = -5$$

Characteristic equation:

$$4\lambda^2 - 25 = 0$$

$$(2\lambda_1 - 5)(2\lambda_2 + 5) = 0 \quad \Rightarrow \lambda_1 = 2.5, \quad \lambda_2 = -2.5$$

The characteristic equation has two different real roots.

$$\Rightarrow \quad y(x) = c_1 e^{2.5x} + c_2 e^{-2.5x}$$

$$and\ \ y'(x) = 2.5c_1 e^{2.5x} - 2.5c_2 e^{-2.5x}$$

Initial values give:

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = 2.5c_1 - 2.5c_2 = -5 \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 1 \end{cases}$$

Therefore,

$$y(x) = -e^{2.5x} + e^{-2.5x}$$

Check:

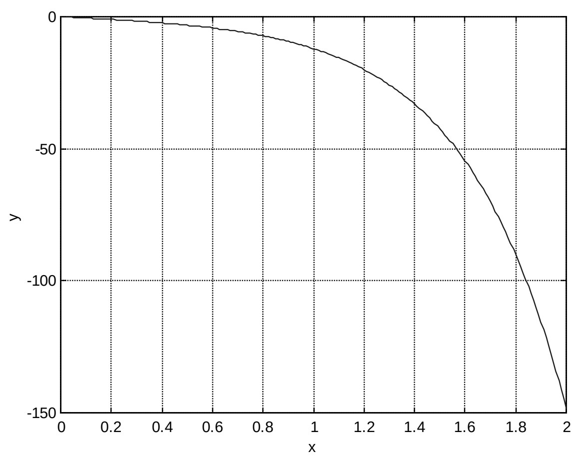
$$y'(x) = -2.5e^{2.5x} - 2.5e^{-2.5x}$$

$$y''(x) = -6.25e^{2.5x} + 6.25e^{-2.5x}$$

$$\begin{aligned}\therefore 4y'' - 25y &= 4(-6.25e^{2.5x} + 6.25e^{-2.5x}) - 25(-e^{2.5x} + e^{-2.5x}) \\ &= (-25+25)e^{2.5x} + (25-25)e^{-2.5x} \\ &= 0\end{aligned}$$

$$and\ \ y(0) = -1+1 = 0, \quad y'(0) = -2.5 - 2.5 = -5$$

Plot with Matlab:



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1. Problem 15 (plot y(x) between 0 and 2)

$$y'' - 25y = 0, \quad y(0) = 0, \quad y'(0) = 20$$

Characteristic equation:

$$\lambda^2 - 25 = 0$$

$$(\lambda_1 - 5)(\lambda_2 + 5) = 0 \quad \Rightarrow \lambda_1 = 5, \quad \lambda_2 = -5$$

The characteristic equation has two different real roots.

$$\Rightarrow \quad y(x) = c_1 e^{5x} + c_2 e^{-5x}$$

$$and\ \ y'(x) = 5c_1 e^{5x} - 5c_2 e^{-5x}$$

Initial values give:

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = 5c_1 - 5c_2 = 20 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -2 \end{cases}$$

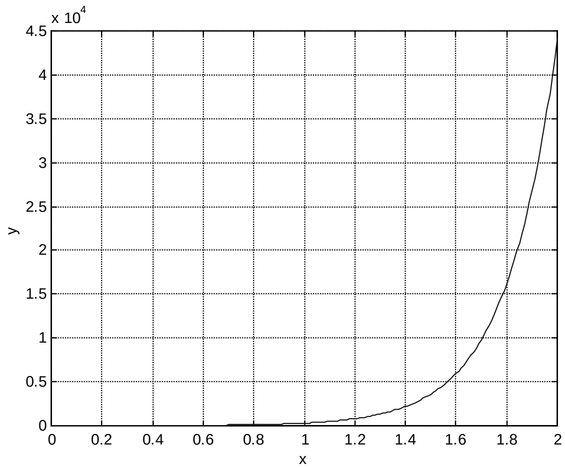
Therefore,

$$y(x) = 2e^{5x} - 2e^{-5x}$$

Check:

$$y'(x) = 10e^{5x} + 10e^{-5x}$$
$$y''(x) = 50e^{5x} - 50e^{-5x}$$
$$\therefore\ y'' - 25y = (50e^{5x} - 50e^{-5x}) - 25(2e^{5x} - 2e^{-5x})$$
$$= (50-50)e^{5x} + (-50+50)e^{-5x}$$
$$= 0$$
$$and\ \ y(0) = 2 - 2 = 0, \quad y'(0) = 10 + 10 = 20$$

Plot with Matlab:



2. Problem 17 (plot y(x) between 0 and 2)

$$y'' - y' - 2y = 0, \quad y(0) = -4, \quad y'(0) = -17$$

Characteristic equation:

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda_1 - 2)(\lambda_2 + 1) = 0 \quad \Rightarrow \lambda_1 = 2, \quad \lambda_2 = -1$$

The characteristic equation has two different real roots.

$$\Rightarrow \quad y(x) = c_1 e^{2x} + c_2 e^{-x}$$
$$and\ \ y'(x) = 2c_1 e^{2x} - c_2 e^{-x}$$

Initial values give:

$$\begin{cases} y(0) = c_1 + c_2 = -4 \\ y'(0) = 2c_1 - c_2 = -17 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = -7 \\ c_2 = 3 \end{cases}$$

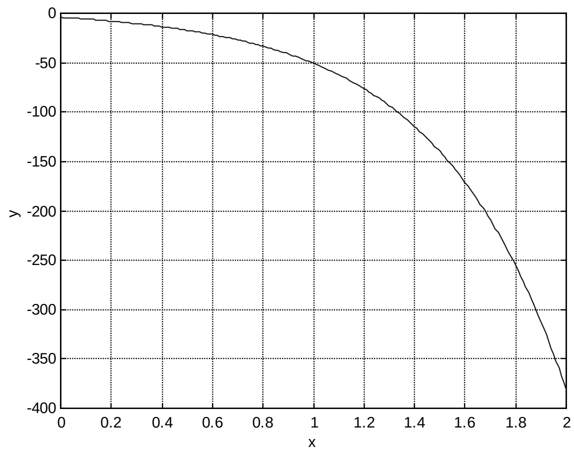
Therefore,

$$y(x) = -7e^{2x} + 3e^{-x}$$

Check:

$$y'(x) = -14e^{2x} - 3e^{-x}$$
$$y''(x) = -28e^{2x} + 3e^{-x}$$
$$\therefore\ y'' - y' - 2y = (-28e^{2x} + 3e^{-x}) - (-14e^{2x} - 3e^{-x}) - 2(-7e^{2x} + 3e^{-x})$$
$$= (-28 + 14 + 14)e^{2x} + (3 + 3 - 6)e^{-x}$$
$$= 0$$
$$and\ \ y(0) = -7 + 3 = -4, \quad y'(0) = -14 - 3 = -17$$

Plot with Matlab:



3. Problem 19 (plot y(x) between 0 and π/2)

$$y'' + 4y = 0, \quad y(0) = 3, \quad y(\tfrac{\pi}{2}) = -3$$

Characteristic equation:

$$\lambda^2 + 4 = 0$$

$$(\lambda_1 - 2i)(\lambda_2 + 2i) = 0 \quad \Rightarrow \lambda_1 = 2i, \quad \lambda_2 = -2i$$

$$i.e. \quad a = 0, \omega = 2$$

The characteristic equation has two complex roots.

$$\Rightarrow \quad y(x) = e^{-\frac{ax}{2}}(A\cos\omega x + B\sin\omega x)$$
$$= A\cos 2x + B\sin 2x$$

Boundary values give:

$$\begin{cases} y(0) = A = 3 \\ y(\tfrac{\pi}{2}) = -A = -3 \end{cases} \Rightarrow \begin{cases} A = 3 \\ B\ undecided \end{cases}$$

So we see that these boundary values yield no condition for B. This problem has actually infinite solutions. Therefore,

$$y(x) = 3\cos 2x + B\sin 2x$$

where B is arbitrary.

Check:

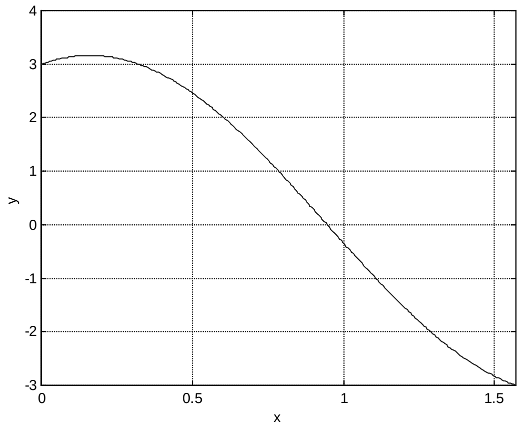
$$y'(x) = -6\sin 2x + 2B\cos 2x$$

$$y''(x) = -12\cos 2x - 4B\sin 2x$$

$$\therefore y'' + 4y = (-12\cos 2x - 4B\sin 2x) + 4(3\cos 2x + B\sin 2x)$$
$$= (-12 + 12)\cos 2x + (-4B + 4B)\sin 2x$$
$$= 0$$

$$and \ y(0) = 3, \quad y(\tfrac{\pi}{2}) = -3$$

Plot with Matlab (assume B=1):


4
3
2
1
0
-1
-2
-3
y
0
0.5
1
1.5
x