Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

Problem Set 2 – Solution

Section 12.4 p. 673

1. Problem 3

$$f(z) = e^x(\cos y + i\sin y)$$
$$u = \mathrm{Re}[f(z)] = e^x \cos y, \quad v = \mathrm{Im}[f(z)] = e^x \sin y$$
$$u_x = e^x \cos y, \quad v_x = e^x \sin y$$
$$u_y = -e^x \sin y, \quad v_y = e^x \cos y$$
$$\therefore u_x = v_y, \quad and \quad u_y = -v_x$$

Therefore, by the Cauchy-Riemann condition, *f(z)* is analytic.

2. Problem 5

$$f(z) = z\bar{z} = (x+iy)(x-iy) = x^2 + y^2$$
$$u = \mathrm{Re}[f(z)] = x^2 + y^2, \quad v = \mathrm{Im}[f(z)] = 0$$
$$u_x = 2x, \quad v_x = 0$$
$$u_y = 2y, \quad v_y = 0$$
$$\therefore u_x \neq v_y, \quad and \quad u_y \neq -v_x$$

Therefore, the Cauchy-Riemann condition is not satisfied for all *z* and *f(z)* is not analytic.

3. Problem 8

$$f(z) = \frac{1}{1-z^4} = \frac{1}{1-r^4(\cos\theta + i\sin\theta)^4} = \frac{1-r^4\cos 4\theta + i\sin 4\theta}{1-2r^4\cos 4\theta + r^8}$$
$$u = \mathrm{Re}[f(z)] = \frac{1-r^4\cos 4\theta}{1-2r^4\cos 4\theta + r^8}, \quad v = \mathrm{Im}[f(z)] = \frac{\sin 4\theta}{1-2r^4\cos 4\theta + r^8}$$
$$u_r = \frac{4r^3\left[\cos 4\theta + r^8\cos 4\theta - 2r^4\right]}{\left(1-2r^4\cos 4\theta + r^8\right)^2}, \quad u_\theta = \frac{4r^4\sin 4\theta(r^8-1)}{\left(1-2r^4\cos 4\theta + r^8\right)^2}$$
$$v_r = -\frac{4r^3\sin 4\theta(r^8-1)}{\left(1-2r^4\cos 4\theta + r^8\right)^2}, \quad v_\theta = \frac{4r^4\left[\cos 4\theta + r^8\cos 4\theta - 2r^4\right]}{\left(1-2r^4\cos 4\theta + r^8\right)^2}$$
$$\therefore u_r = \frac{1}{r}v_\theta, \quad and \quad v_r = -\frac{1}{r}u_\theta$$

Therefore, by the Cauchy-Riemann condition, *f(z)* is analytic. Strictly speaking, the domain over which *f(z)* is analytic should exclude points z = ±1 and ±*i* since at these points *f(z)* blows up and the Cauchy-Riemann condition cannot be applied.

4. Problem 11

$$f(z) = \mathrm{Re}(z^3) = \mathrm{Re}\left((x+iy)^3\right) = \mathrm{Re}\left(x^3 - iy^3 + i3x^2y - 3xy^2\right) = x^3 - 3xy^2$$

$$u = \mathrm{Re}[f(z)] = x^3 - 3xy^2, \quad v = \mathrm{Im}[f(z)] = 0$$

$$u_x = 3x^2 - 3y^2, \quad u_y = -6xy$$

$$v_x = 0, \quad v_y = 0$$

$$\therefore u_x \neq v_y, \quad and \quad u_y \neq -v_x$$

Therefore, the Cauchy-Riemann condition is not satisfied for all *z* and *f(z)* is not analytic.

Section 12.6 p. 682

1. Problem 1

$$e^z = e^{2+3\pi i} = e^2 e^{3\pi i} = e^2(\cos 3\pi + i\sin 3\pi) = -e^2 = -7.3891$$

$$\left|e^z\right| = \left|e^{2+3\pi i}\right| = 7.3891$$

2. Problem 6

$$1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1.414\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1.414e^{i\pi/4}$$

3. Problem 13 (you don't need to plot)

$$e^z = e^{x+iy} = -3 = 3e^{i(\pi \pm 2n\pi)} \qquad (n = 0,1,2,....)$$

$$where$$

$$e^x = 3 \quad \Rightarrow \quad x = \ln 3 = 1.0986$$

$$y = \pi \pm 2n\pi$$

$$\therefore z = 1.0986 + i(\pi \pm 2n\pi) \qquad (n = 0,1,2,....)$$

4. Problem 15 (you don't need to plot)

From (11) of this section, we know that, in any case,

$$e^z \neq 0$$

So there is no solution to this problem.

Section 12.7 p. 686-687

1. Problem 7

$$\cosh(-3-6i) = \frac{1}{2}(e^{-3-6i} + e^{3+6i})$$

$$= 0.5\left\{e^{-3}\left[\cos(-6) + i\sin(-6)\right] + e^3(\cos 6 + i\sin 6)\right\}$$

$$= 9.6667 - 2.7991i$$

or,

$$\cosh(-3-6i) = \cosh(-3)\cos(-6) + i\sinh(-3)\sin(-6) = 9.6667 - 2.7991i$$

2. Problem 8

$$\sinh(4+5i)=\frac{1}{2}\left(e^{4+5i}-e^{-4-5i}\right)$$
$$=0.5\left\{e^{4}(\cos 5+i\sin 5)-e^{-4}\left[\cos(-5)+i\sin(-5)\right]\right\}$$
$$=7.7411-26.1865i$$

or,

$$\sinh(4+5i)=\sinh 4\cos 5+i\cosh 4\sin 5=7.7411-26.1865i$$

3. Problem 9

$$\cos z=\frac{1}{2}(e^{iz}+e^{-iz})=3i$$
$$\therefore e^{2iz}-6ie^{iz}+1=0$$
$$\Rightarrow e^{iz}=e^{-y+xi}=\frac{6i\pm\sqrt{-36-4}}{2}=6.1623i \quad and \quad -0.1623i$$
$$thus\, e^{-y}=6.1623, e^{xi}=i \quad \Rightarrow \quad y=-1.818\, and\;\, x=\frac{\pi}{2}\pm 2n\pi \quad (n=0,1,2...)$$
$$or\, e^{-y}=-0.1623, e^{xi}=-i \quad \Rightarrow \quad y=1.818\, and\;\, x=-\frac{\pi}{2}\pm 2n\pi \quad (n=0,1,2...)$$
$$\therefore z=\frac{\pi}{2}\pm 2n\pi-1.818i \quad or \quad -\frac{\pi}{2}\pm 2n\pi+1.818i \quad (n=0,1,2...)$$

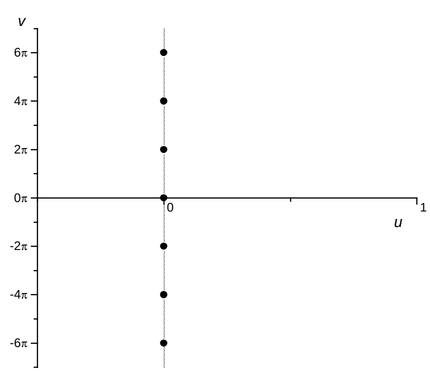
Section 12.8 p. 691-692

1. Problem 7 (just do ***1 + i*** here)

$$Ln(1+i)=\ln|1+i|+iArg(1+i)=0.3466+i\pi/4$$

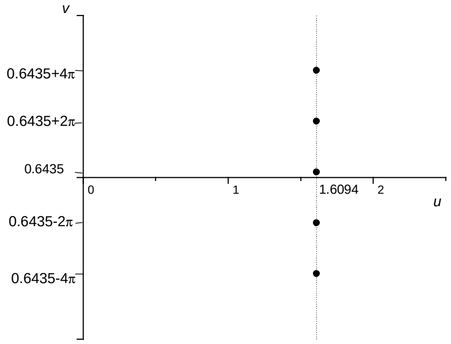
2. Problem 11 (you don't need to plot)

$$\ln 1=0,\pm 2\pi i,\pm 4\pi i,\ldots$$



3. Problem 14 (you don't need to plot)

$$\ln(4+3i) = \ln|4+3i| + i\arg(4+3i)$$
$$= 1.6094 + (0.6435 \pm 2n\pi)i \quad (n = 0,1,2,\ldots)$$



4. Problem 17

$$\ln z = 4 - 3i$$
$$z = e^{4-3i} = e^{4}(\cos(-3) + i\sin(-3)) = -54.0518 - 7.7049i$$

5. Problem 23 (this is ***1 + i*** raised to the ***-1 + i*** power)

$$(1+i)^{-1+i} = \exp[(-1+i)Ln(1+i)] = \exp\left[(-1+i)\left(\ln\sqrt{2} + i\frac{\pi}{4}\right)\right]$$
$$= \exp\left[-\ln\sqrt{2} - \frac{\pi}{4} + i\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right]$$
$$= 0.2919 - 0.1370i$$