Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering
Problem Set 1 – Solution

<u>Section 12.1 p. 657</u>

1. Problem 5

$$\frac{1}{z_1} = \frac{1}{4+3i} = \frac{4-3i}{(4+3i)(4-3i)} = \frac{4-3i}{16+9} = \frac{4}{25} - \frac{3}{25}i$$

2. Problem 7

$$\begin{aligned} \mathrm{Re}({z_1}^3) &= \mathrm{Re}[(4+3i)^3] \\ &= \mathrm{Re}[4^3 + 3\cdot 4^2 \cdot (3i) + 3\cdot 4\cdot (3i)^2 + (3i)^3] \\ &= \mathrm{Re}[64 + 144i - 108 - 27i] \\ &= 64 - 108 \\ &= -44 \end{aligned}$$

$$[\mathrm{Re}(z1)]^3 = [\mathrm{Re}(4+3i)]^3 = 4^3 = 64$$

3. Problem 8

$$\begin{aligned} \frac{z1-z2}{z1+z2} &= \frac{(4+3i)-(2-5i)}{(4+3i)+(2-5i)} \\ &= \frac{2+8i}{6-2i} = \frac{1+4i}{3-i} \\ &= \frac{(1+4i)(3+i)}{(3-i)(3+i)} = \frac{3+i+12i-4}{9+1} \\ &= -\frac{1}{10} + \frac{13}{10}i \end{aligned}$$

4. Problem 10

$$\begin{aligned} \frac{1}{{z_1}^2} &= \frac{1}{(4+3i)^2} = \frac{1}{16+24i-9} = \frac{1}{7+24i} \\ &= \frac{(7-24i)}{(7+24i)(7-24i)} = \frac{7-24i}{49+576} \\ &= \frac{7}{625} - \frac{24}{625}i \end{aligned}$$

$$\frac{1}{\bar{z}_1^{\,2}} = \frac{1}{\overline{z_1^{\,2}}} = \overline{\left(\frac{1}{z_1^{\,2}}\right)} = \overline{\left(\frac{7}{625} - \frac{24}{625}i\right)} = \frac{7}{625} + \frac{24}{625}i$$

5. Problem 12

$$\frac{z_2\bar{z}_2}{z_1\bar{z}_1} = \frac{(2-5i)(2+5i)}{(4+3i)(4-3i)} = \frac{4+25}{16+9} = \frac{29}{25}$$

6. Problem 17

$$\mathrm{Re}(z^2/\bar{z}) = \mathrm{Re}\left(\frac{(x+iy)^2}{(x-iy)}\right) = \mathrm{Re}\left(\frac{(x+iy)^3}{(x-iy)(x+iy)}\right)$$

$$= \mathrm{Re}\left(\frac{x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3}{x^2+y^2}\right)$$

$$= \frac{x^3 - 3xy^2}{x^2+y^2}$$

Section 12.2 p. 662

1. Problem 3

$$z = -3 - 4i$$

$$|z| = r = \sqrt{(-3)^2 + (-4)^2} = 5$$

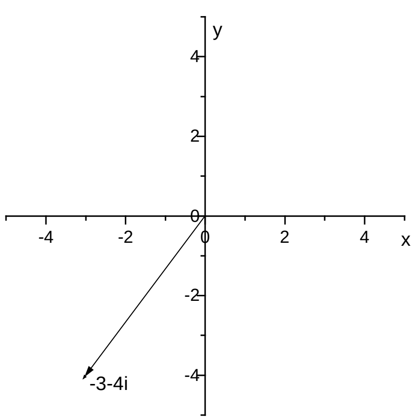
$$\arg z = \tan^{-1}\left(\frac{-4}{-3}\right) = 0.927 \quad or \quad -2.214$$

For any argument, the inverse tangent can take two values in the range (-π, π]. The correct value is decided by looking at the quadrant where z lies.

Re(-3-4i) < 0, Im(-3-4i) < 0

So, z lies in the third quadrant. Therefore $\arg z = -2.214$ and

$$z = -3 - 4i = 5[\cos(-2.214) + i\sin(-2.214)]$$



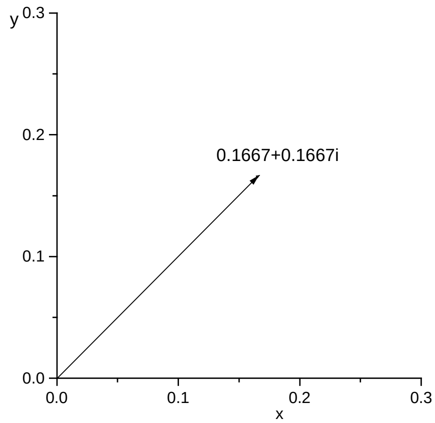
2. Problem 8

$$z = \frac{i}{3+3i}$$

$$|z| = \left|\frac{i}{3+3i}\right| = \frac{|i|}{|3+3i|} = \frac{1}{\sqrt{3^2+3^2}} = 0.2357$$

$$\arg z = \arg(i) - \arg(3+3i) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\therefore \quad z = 0.2357(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 0.1667 + 0.1667i$$



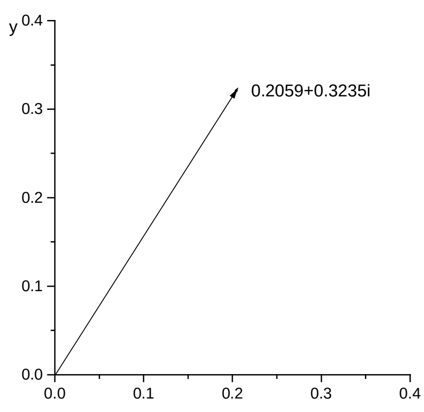
3. Problem 9

$$z = \frac{2+i}{5-3i}$$

$$|z| = \left|\frac{2+i}{5-3i}\right| = \frac{|2+i|}{|5-3i|} = \frac{\sqrt{2^2+1}}{\sqrt{5^2+(-3)^2}} = 0.3835$$

$$\arg z = \arg(2+i) - \arg(5-3i) = 1.004$$

$$\therefore \quad z = 0.3835(\cos 1.004 + i\sin 1.004) = 0.2059 + 0.3235i$$



4. Problem 14

$$z = -5 + 5i$$

$$\arg z = \tan^{-1}\left(\frac{5}{-5}\right) = -\frac{\pi}{4} \quad or \quad \frac{3\pi}{4}$$

For any argument, the inverse tangent can take two values in the range (-π, π]. The correct value is decided by looking at the quadrant where z lies.

Re(-5+5i) < 0, Im(-5+5i) > 0

So, z lies in the second quadrant. Therefore

$$\arg z = \frac{3\pi}{4}$$

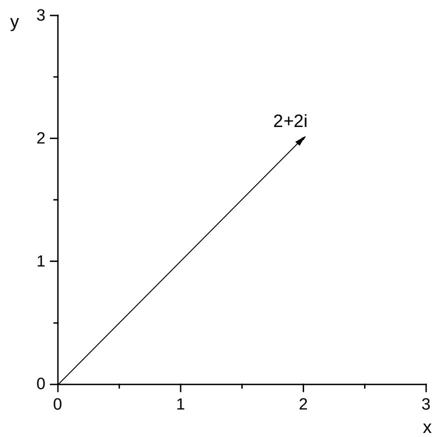
5. Problem 17

$$z = \sqrt{8}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = x + iy$$

$$where \quad x = \sqrt{8}\cos\frac{\pi}{4} = \sqrt{8}\cdot\frac{\sqrt{2}}{2} = 2$$

$$y = \sqrt{8}\sin\frac{\pi}{4} = \sqrt{8}\cdot\frac{\sqrt{2}}{2} = 2$$

$$\therefore z = 2 + 2i$$



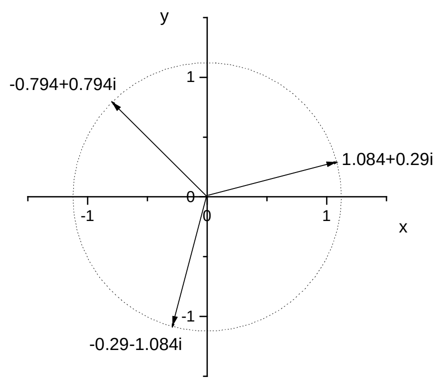
6. Problem 21

$$\sqrt[3]{1+i} = \left[\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})\right]^{\frac{1}{3}}$$

$$= \sqrt[6]{2}\left(\cos\frac{\pi/4 + 2k\pi}{3} + i\sin\frac{\pi/4 + 2k\pi}{3}\right) \qquad k = 0,1,2$$

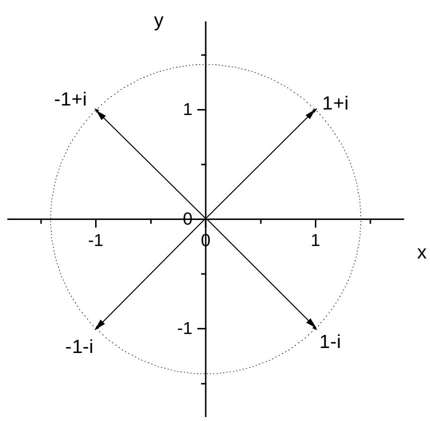
$$= \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \quad \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \quad \sqrt[6]{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

$$= 1.084 + 0.290i, \quad -0.794 + 0.794i, \quad -0.290 - 1.084i$$



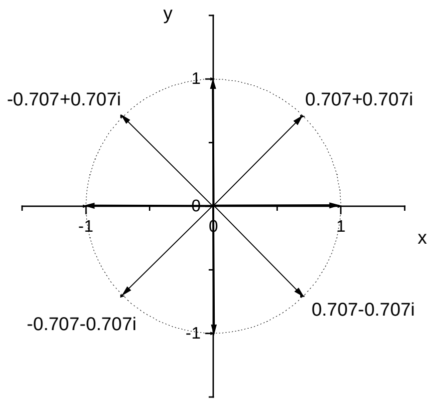
7. Problem 24

$$\sqrt[4]{-4} = [4(\cos\pi + i\sin\pi)]^{\frac{1}{4}}$$
$$= \sqrt{2}\left(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}\right) \qquad k = 0,1,2,3$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \quad \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \quad \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \quad \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$
$$= 1+i, \quad -1+i, \quad -1-i, \quad 1-i$$



8. Problem 27

$$\sqrt[8]{1} = [\cos 0 + i\sin 0]^{\frac{1}{8}}$$
$$= \left(\cos\frac{2k\pi}{8} + i\sin\frac{2k\pi}{8}\right) \qquad k = 0,1,\cdots 7$$
$$= 1, \quad 0.707 + 0.707i, \quad i, \quad -0.707 + 0.707i,$$
$$-1, \quad -0.707 - 0.707i, \quad -i, \quad 0.707 - 0.707i$$



Section 12.3 p. 668

1. Problem 11

$$f = \frac{1}{1-z}$$

$$f(7+2i) = \frac{1}{1-7-2i} = \frac{-6+2i}{(-6-2i)(-6+2i)}$$

$$= \frac{-6+2i}{40} = -0.15 + 0.05i$$

$$\therefore \operatorname{Re}(f(7+2i)) = -0.15 \quad and \quad \operatorname{Im}(f(7+2i)) = 0.05$$

(In these next problems, just take the derivative with respect to ***z*** and plug in the point given to solve the problem. We'll see how to check functions like this for the existence of their derivatives soon, but for now, you can just take our word for it that the derivative exists.)

2. Problem 19 (derivative exists, no need to check)

$$f'(z) = \frac{d}{dz}\left(\frac{5+3i}{z^3}\right) = \frac{-3(5+3i)}{z^4} = \frac{-15-9i}{z^4}$$

$$\therefore\ f'(2+i) = \frac{-15-9i}{(2+i)^4} = \frac{-15-9i}{-7+24i} = \frac{(-15-9i)(-7-24i)}{(-7+24i)(-7-24i)}$$

$$= \frac{-111+423i}{625} = -0.1776 + 0.6768i$$

3. Problem 20 (derivative exists, no need to check)

$$f'(z) = \frac{d}{dz}\left[(3z^2 + iz)^2\right] = 2(3z^2 + iz)(6z + i) = 36z^3 + i18z^2 - 2z$$

$$\therefore\ f'(1+i) = 36(1+i)^3 + i18(1+i)^2 - 2(1+i)$$

$$= -110 + 70i$$

4. Problem 21 (derivative exists, no need to check)

$$f'(z) = \frac{d}{dz}\left[z^4 + \frac{1}{z^4}\right] = 4z^3 - 4z^{-5}$$

$$\therefore\ f'(-1-i) = 4(-1-i)^3 - 4(-1-i)^{-5}$$

$$= 7.5 - 7.5i$$