Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

FINAL EXAM – Solution

Problem 1 (8 points):

Find 5 distinct values for:

$$\sqrt[5]{-j64}$$

$$\sqrt[5]{-j64} = \left[64(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2})\right]^{\frac{1}{5}}$$

$$= \sqrt[5]{64}(\cos\frac{\frac{3\pi}{2}+2k\pi}{5} + j\sin\frac{\frac{3\pi}{2}+2k\pi}{5}) \qquad (k = 0,1,2,3,4)$$

$$= \sqrt[5]{64}(\cos\frac{3\pi}{10} + j\sin\frac{3\pi}{10}),\quad \sqrt[5]{64}(\cos\frac{7\pi}{10} + j\sin\frac{7\pi}{10}),\quad \sqrt[5]{64}(\cos\frac{11\pi}{10} + j\sin\frac{11\pi}{10})$$

$$\sqrt[5]{64}(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}),\quad \sqrt[5]{64}(\cos\frac{19\pi}{10} + j\sin\frac{19\pi}{10})$$

Problem 2 (8 points):

Evaluate the expression below. Give the principal value only in x+jy notation.

$$z = \left(-1 - j\sqrt{3}\right)^{(1+j)}$$

$$z = \left(-1 - j\sqrt{3}\right)^{(1+j)} = e^{\ln\left[\left(-1-j\sqrt{3}\right)^{(1+j)}\right]} = e^{(1+j)\ln\left(-1-j\sqrt{3}\right)}$$

$$= e^{(1+j)\ln\left(2e^{-j\frac{2\pi}{3}}\right)} = e^{(1+j)\left(\ln 2 - j\frac{2\pi}{3}\right)} = e^{\left(\ln 2 + \frac{2\pi}{3} + j(\ln 2 - \frac{2\pi}{3})\right)}$$

$$= e^{(2.78754 - j1.40125)}$$

$$= 2.7405 - j16.0082$$

Problem 3 (10 points):

For the complex function:

$f(z) = z^2 + e^z$

a) find *u(x,y)* and *v(x,y)* for *f(z=x+jy) = u(x,y) + j v(x,y)*

$$f(x+jy) = (x+jy)^2 + e^{(x+jy)}$$
$$= (x^2 - y^2 + j2xy) + e^x(\cos y + j\sin y)$$
$$= (x^2 - y^2 + e^x \cos y) + j(2xy + e^x \sin y)$$
$$\therefore \quad u(x,y) = x^2 - y^2 + e^x \cos y, \quad v(x,y) = 2xy + e^x \sin y$$

b) show that f(z) is analytic for all values of z.

$$u_x = 2x + e^x \cos y, \quad u_y = -2y - e^x \sin y$$
$$v_x = 2y + e^x \sin y, \quad v_y = 2x + e^x \cos y$$
$$so \qquad u_x = v_y, \quad u_y = -v_x$$

f(z) is analytic for all values of z.

c) find the value of:

i) f(z) evaluated at the point (z = -jπ/2)

$$f(-j\frac{\pi}{2}) = \left(-j\frac{\pi}{2}\right)^2 + e^{-j\frac{\pi}{2}} = -\frac{\pi^2}{4} + \left[\cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right]$$
$$= -\frac{\pi^2}{4} - j$$
$$= -2.4674 - j$$

ii) df/dz evaluated at the point (z = -jπ/2)

$$\frac{df}{dz} = 2z + e^z$$
$$\left.\frac{df}{dz}\right|_{z=-j\frac{\pi}{2}} = 2\left(-j\frac{\pi}{2}\right) + e^{-j\frac{\pi}{2}} = -j\pi + j\sin\left(-\frac{\pi}{2}\right) = -j(\pi+1) = -4.1416j$$

Problem 4 (10 points):

Find the solution, y(t) for the following second-order linear differential equation and initial conditions.

$1/8\ y'' + y' + 2\ y = 16t^2$ $\qquad$ $y(0) = 4$ $\qquad$ $y'(0) = -23/2$

$$\frac{1}{8}y'' + y' + 2y = 16t^2, \quad y(0) = 4, \quad y'(0) = -\frac{23}{2}$$

$$y = y_p + y_h$$

Particular solution:

$$\frac{1}{8}y_p'' + y_p' + 2y_p = 16t^2 \qquad \text{Choose} \quad y_p = a_0 + a_1 t + a_2 t^2$$

$$y_p' = a_1 + 2a_2 t, \quad y_p'' = 2a_2$$

$$\frac{1}{8}y_p'' + y_p' + 2y_p = \frac{1}{4}a_2 + a_1 + 2a_2 t + 2\left(a_0 + a_1 t + a_2 t^2\right)$$

$$= \left(\frac{1}{4}a_2 + a_1 + 2a_0\right) + (2a_2 + 2a_1)t + 2a_2 t^2$$

$$\Rightarrow \begin{cases} \frac{1}{4}a_2 + a_1 + 2a_0 = 0 \\ 2a_2 + 2a_1 = 0 \\ 2a_2 = 16 \end{cases} \Rightarrow \begin{cases} a_0 = 3 \\ a_1 = -8 \\ a_2 = 8 \end{cases}$$

$$\Rightarrow y_p = 8t^2 - 8t + 3, \qquad y_p(0) = 3, \quad y_h(0) = y(0) - y_p(0) = 4 - 3 = 1$$

$$y_p' = 16t - 8, \qquad y_p'(0) = -8, \quad y_h'(0) = y'(0) - y_p'(0) = -\frac{23}{2} - (-8) = -\frac{7}{2}$$

Homogeneous solution:

$$\frac{1}{8}y_h'' + y_h' + 2y_h = 0$$

$$\frac{1}{8}\lambda^2 + \lambda + 2 = 0$$

$$\lambda_1 = \lambda_2 = -4$$

$$y_h = c_1 e^{-4t} + c_2 t e^{-4t}$$

$$y_h' = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t},$$

$$\begin{cases} y_h(0) = c_1 = 1 \\ y_h'(0) = -4c_1 + c_2 = -\frac{7}{2} \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = \frac{1}{2} \end{cases}$$

$$y_h = e^{-4t} + \frac{1}{2}te^{-4t}$$

$$\therefore \quad y(t) = 8t^2 - 8t + 3 + e^{-4t} + \frac{1}{2}te^{-4t}$$

Problem 5 (8 points):

Find the **determinant** and **matrix inverse**, of the following matrix **A**:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 0 + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \qquad \textit{Expand by } 1\textit{st column}$$

$$= (3-2) + 0 + (1-1)$$

$$= 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad R3 \Leftarrow R3 - R2 - R1$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad R1 \Leftarrow R1 - R2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad R2 \Leftarrow R2 - R3$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Problem 6 (16 points):

Solve the following set of differential equations with initial conditions by:
(a) (10 points) Identifying the system matrix **A** and finding the eigenvalues and eigenvectors for the **A**.

(b) (6 points) Use the eigenvalues and eigenvectors from (a) along with the initial conditions to solve for the vector **x**.

$dx_1(t)/dt = 3\,x_1(t);$ $\quad x_1(0) = -3$
$dx_2(t)/dt = 3\,x_2(t) + 4\,x_3(t));$ $\quad x_2(0) = 3$
$dx_3(t)/dt = 4\,x_2(t) + 3\,x_3(t);$ $\quad x_3(0) = 5$

(a)

The system matrix A :

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \qquad y' = Ay \quad y(0) = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$= (3-\lambda)[(3-\lambda)^2 - 16]$$
$$= (3-\lambda)(7-\lambda)(-1-\lambda)$$

$$\therefore \quad \lambda_1 = 3, \quad \lambda_2 = 7, \quad \lambda_3 = -1$$

Eigenvectors:

$\lambda_1 = 3$

$(A - \lambda_1 I)x_1 = 0$

$$A - \lambda_1 I = \begin{bmatrix} 3-\lambda_1 & 0 & 0 \\ 0 & 3-\lambda_1 & 4 \\ 0 & 4 & 3-\lambda_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow x_{12} = 0, \quad x_{13} = 0$$

$$Choose \;\; x_{11} = 1, \qquad x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 7$

$(A - \lambda_2 I)x_2 = 0$

$$A - \lambda_2 I = \begin{bmatrix} 3-\lambda_2 & 0 & 0 \\ 0 & 3-\lambda_2 & 4 \\ 0 & 4 & 3-\lambda_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow x_{21} = 0, \quad x_{22} = x_{23}$$

Choose $x_{22} = 1, \qquad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = -1$

$(A - \lambda_3 I)x_3 = 0$

$$A - \lambda_3 I = \begin{bmatrix} 3-\lambda_3 & 0 & 0 \\ 0 & 3-\lambda_3 & 4 \\ 0 & 4 & 3-\lambda_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow x_{31} = 0, \quad x_{32} = -x_{33}$$

Choose $x_{32} = 1, \qquad x_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

b)

$$y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} + c_3 x_3 e^{\lambda_3 t}$$

$$y(0) = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$$

$$\therefore \quad y = -3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{7t} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$$

Problem 7 (8 points):

A volume charge density ρ(r, z) is distributed within the volume of a cylinder of radius R and height H that sits at z = 0 (i.e., in the horizontal x-y plane which cuts through the origin of the z-axis). The functional dependence of ρ(r, z) upon position within the volume is:

$$\rho(r, z) = \rho_o \left[1 - \frac{rz}{RH} \right]$$

where ρ_o is a known constant with units C/m^3.

Calculate the total charge Q within the cylinder.

$$Q = \iiint_V \rho(r, z) dV$$

$$= \int_{\phi=0}^{2\pi} \int_{z=0}^{H} \int_{r=0}^{R} \rho_0 \left[1 - \frac{rz}{RH} \right] r d\phi \, dz dr$$

$$= 2\pi\rho_0 \left[\int_{r=0}^{R} r dr \int_{z}^{H} dz - \frac{1}{RH} \int_{r=0}^{R} r^2 dr \int_{z=0}^{H} z dz \right]$$

$$= 2\pi\rho_0 \left\{ \left[\frac{r^2}{2} \right]_{r=0}^{R} \cdot [z]_{z=0}^{H} - \frac{1}{RH} \left[\frac{r^3}{3} \right]_{r=0}^{R} \cdot \left[\frac{z^2}{2} \right]_{z=0}^{H} \right\}$$

$$= 2\pi\rho_0 \left(\frac{R^2 H}{2} - \frac{R^2 H}{6} \right)$$

$$= \frac{2}{3}\pi\rho_0 R^2 H$$

Problem 8 (8 points):

A surface charge density σ(θ) is distributed over the surface of a hemisphere with radius R. The base of the hemisphere is sitting in the x-y plane at z = 0. The functional dependence of σ(θ) upon angular position with respect to the vertical axis is:

$$\sigma(\theta) = \sigma_o \cos^3\theta$$

where σ_o is a known constant with units C/m^2.

Find the total charge Q on the top surface of the hemisphere (**exclude** the base).

$$Q = \iint_s \sigma(\theta) ds$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \sigma_0 \cos^3\theta \, R^2 \sin\theta \, d\theta \, d\phi$$

$$= \sigma_0 R^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta$$

$$= -2\pi\sigma_0 R^2 \int_{\theta=0}^{\frac{\pi}{2}} \cos^3\theta \, d(\cos\theta) \qquad \sin\theta = -d(\cos\theta)$$

$$= -2\pi\sigma_0 R^2 \left(\frac{\cos^4\theta}{4} \right)_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{\pi\sigma_0 R^2}{2}$$

Problem 9 (8 points):

a) For any general scalar field ϕ(x, y, z), show that:

$$\nabla \mathbf{x}(\nabla\phi) = 0$$

(where the "**x**" in this equation is the "cross product")

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{a}_x + \frac{\partial\phi}{\partial y}\vec{a}_y + \frac{\partial\phi}{\partial z}\vec{a}_z$$

$$\nabla\times(\nabla\phi) = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{a}_x\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] + \vec{a}_y\left[\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \vec{a}_z\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= \vec{a}_x 0 + \vec{a}_y 0 + \vec{a}_z 0$$

$$= 0$$

b) For any general vector field **B**(x, y, z) given by:

$$\vec{B}(x, y, z) = B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z$$

Show that:

$$\nabla \bullet (\nabla \mathbf{x}\vec{B}) = 0$$

$$\nabla\times\vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \vec{a}_x\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right] + \vec{a}_y\left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right] + \vec{a}_z\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]$$

$$\nabla\cdot(\nabla\times\vec{B}) = \frac{\partial}{\partial x}\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]$$

$$= \frac{\partial^2 B_z}{\partial x\partial y} - \frac{\partial^2 B_y}{\partial x\partial z} + \frac{\partial^2 B_x}{\partial y\partial z} - \frac{\partial^2 B_z}{\partial y\partial x} + \frac{\partial^2 B_y}{\partial z\partial x} - \frac{\partial^2 B_x}{\partial z\partial y}$$

$$= 0$$

Problem 10 (8 points):

Given the vector field **F(x,y,z)** :

$$\vec{F}(x,y,z) = F_o \frac{x^2 y^2}{a^4} \vec{\mathbf{a}}_\mathbf{x}$$

a) Find the *curl* of **F**:

$$\nabla x \vec{F}$$

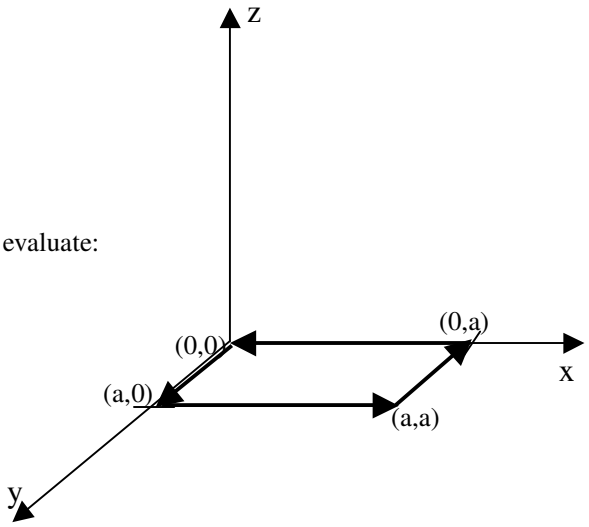
b) For the square surface in the x-y plane below, evaluate:

$$\int_S \left[\nabla x \vec{F}\right] \bullet \vec{a}_n dS$$

OR

$$\oint_C \vec{F} \bullet dl$$

(where **C** is the closed contour in the direction of the arrows around the square)



a)

$$F_x = F_0 \frac{x^2 y^2}{a^4}, \quad F_y = F_z = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & 0 & 0 \end{vmatrix} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_0 \frac{x^2 y^2}{a^4} & 0 & 0 \end{vmatrix} = -F_0 \frac{2x^2 y}{a^4} \vec{a}_z$$

b)

$$\vec{a}_n = \vec{a}_z$$

$$\int_S \left[\nabla \times \vec{F}\right] \cdot \vec{a}_n dS = -\int_{x=0}^{a} \int_{y=0}^{a} F_0 \frac{2x^2 y}{a^4} \vec{a}_z \cdot \vec{a}_z dx dy$$

$$= -\frac{2F_0}{a^4} \int_{x=0}^{a} x^2 dx \int_{y=0}^{a} y dy$$

$$= -\frac{2F_0}{a^4} \left[\frac{x^3}{3}\right]_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{a}$$

$$= -\frac{F_0 a}{3}$$

$$(0,0) \to (a,0) : y = 0, \vec{F} = 0$$

$$(a,0) \to (a,a) : d\vec{l} = \vec{a}_y dy, \vec{F} \cdot d\vec{l} = \vec{F} \cdot \vec{a}_y dy = 0$$

$$(0,a) \to (0,0) : x = 0,\ \vec{F} = 0$$

$$only\ (a,a) \to (0,a) : y = a,\ \vec{F} = F_0 \frac{x^2}{a^2}\vec{a}_x,\ d\vec{l} = -\vec{a}_x dx$$

$$\therefore \vec{F} \cdot d\vec{l} = F_0 \frac{x^2}{a^2}\vec{a}_x \cdot \left(-\vec{a}_x dx\right) = -F_0 \frac{x^2}{a^2} dx \neq 0$$

$$\oint_C \vec{F} \cdot dl = -\int_0^a F_0 \frac{x^2}{a^2} dx = -\frac{F_0}{a^2}\int_0^a x^2 dx = -\frac{F_0}{a^2}\left[\frac{x^3}{3}\right]_0^a = -\frac{F_0 a}{3}$$

$$= \int_S \left[\nabla \times \vec{F}\right] \cdot \vec{a}_n dS$$

Problem 11 (8 points):

For the same vector field F(x, y, z) as the previous problem:

(a) Evaluate the *divergence* of **F**:

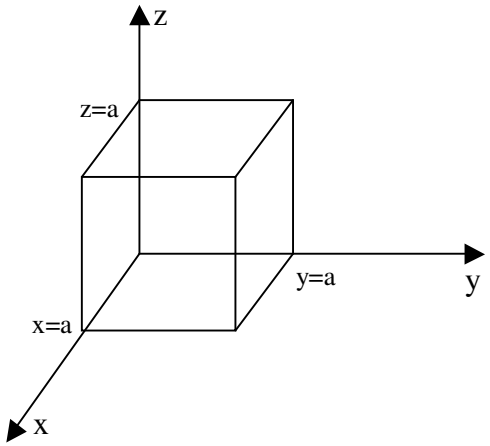
$$\nabla \bullet \vec{F}$$

(b) Evaluate:

$$\int_V \nabla \bullet \vec{F} dV$$

OR

$$\oint_S \vec{F} \bullet \vec{a}_n dS$$



a)

$$F_x = F_0 \frac{x^2 y^2}{a^4}, \quad F_y = F_z = 0$$

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} = F_0 \frac{2xy^2}{a^4}$$

b)

$$\int_V \nabla \cdot \vec{F} dV = \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} F_0 \frac{2xy^2}{a^4} dxdydz$$

$$= \frac{2F_0}{a^4}\int_{x=0}^{a} xdx \int_{y=0}^{a} y^2 dy \int_{z=0}^{a} dz$$

$$= \frac{2F_0}{a^4}\left[\frac{x^2}{2}\right]_{x=0}^{a}\left[\frac{y^3}{3}\right]_{y=0}^{a}[z]_{z=0}^{a}$$

$$= \frac{F_0 a^2}{3}$$

$Surface\ x = 0, y = 0, z = 0 : \vec{F} = 0$

$Surface\ y = a : \vec{a}_n = \vec{a}_y, \vec{F} \cdot \vec{a}_y = 0$

$Surface\ z = a : \vec{a}_n = \vec{a}_z, \vec{F} \cdot \vec{a}_z = 0$

$Surface\ x = a : \vec{a}_n = \vec{a}_x, \vec{F} \cdot \vec{a}_x = F_0 \frac{y^2}{a^2} \neq 0, dS = dydz$

$$\oint_S \vec{F} \cdot \vec{a}_n dS = \int_{y=0}^{a}\int_{z=0}^{a} F_0 \frac{y^2}{a^2} dydz = \frac{F_0}{a^2}\int_{y=0}^{a} y^2 dy \int_{z=0}^{a} dz = \frac{F_0}{a^2}\left[\frac{y^3}{3}\right]_{y=0}^{a}[z]_{z=0}^{a} = \frac{F_0 a^2}{3}$$

$$= \int_V \nabla \cdot \vec{F} dV$$