Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

EXAM 3 – Solution

Problem 1:

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\det A = -1 \cdot \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= -(-4-1) - 6$$

$$= -1$$

$$C_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -5, \quad C_{12} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \quad C_{13} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{21} = \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = 6, \quad C_{22} = \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2, \quad C_{23} = \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3, \quad C_{32} = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{33} = \begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} = -1$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{bmatrix}$$

$$= -1 \cdot \begin{bmatrix} -5 & -6 & 3 \\ -2 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

check:

$$\begin{bmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2:

a)

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 1 & -1 & 1 & 3 \\ 3 & -3 & -1 & 7 \\ -1 & 1 & 1 & 4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & -2 & 17 \\ 0 & 0 & 0 & 12 \end{array}\right] \qquad \textit{Inconstitent}$$

b)

$$\left[\begin{array}{ccc|c} -1 & 3 & 0 & 8 \\ 1 & -2 & 1 & -7 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 3 & 0 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \qquad R2 \Leftarrow R1 + R2$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 3 & 0 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad R3 \Leftarrow R3 - R2$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 3 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad R2 \Leftarrow R2 - R3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad R1 \Leftarrow -R1 + 3R2$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\-2\end{bmatrix}\qquad\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\-2\end{bmatrix}$$

check:

$$\begin{bmatrix}-1&3&0\\1&-2&1\\0&1&2\end{bmatrix}\begin{bmatrix}1\\3\\-2\end{bmatrix}=\begin{bmatrix}8\\-7\\-1\end{bmatrix}$$

Problem 3:

Find the eigenvalues of the matrix:

$$A=\begin{bmatrix}-1&1&0\\1&2&1\\0&3&-1\end{bmatrix}$$

$$|A-\lambda I|=\begin{vmatrix}-1-\lambda&1&0\\1&2-\lambda&1\\0&3&-1-\lambda\end{vmatrix}$$

$$=(-1-\lambda)\begin{vmatrix}2-\lambda&1\\3&-1-\lambda\end{vmatrix}-1\cdot\begin{vmatrix}1&0\\3&-1-\lambda\end{vmatrix}$$

$$=(-1-\lambda)[(2-\lambda)(-1-\lambda)-3]-(-1-\lambda)$$

$$=(-1-\lambda)[-2-2\lambda+\lambda+\lambda^2-3-1]$$

$$=(-1-\lambda)(\lambda^2-\lambda-6)$$

$$=(-1-\lambda)(\lambda+2)(\lambda-3)=0$$

$\lambda_1=-1,\quad \lambda_2=3,\quad \lambda_3=-2$

Eigenvector for $\lambda_1=-1$:

$(A-\lambda_1 I)x_1=0$

$$\left[\begin{array}{ccc|c}-1-\lambda_1&1&0&0\\1&2-\lambda_1&1&0\\0&3&-1-\lambda_1&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}0&1&0&0\\1&3&1&0\\0&3&0&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}0&1&0&0\\1&0&1&0\\0&0&0&0\end{array}\right]$$

$$\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Rightarrow\begin{cases}x_{12}=0\\x_{11}=-x_{13}\end{cases}$$

$$choose\quad x_{11}=1\quad\therefore x_{13}=-1\qquad x_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

check :

$$\begin{bmatrix}-1&1&0\\1&2&1\\0&3&-1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}=(-1)\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Eigenvector for $\lambda_2=3$:

$(A-\lambda_2 I)x_2=0$

$$\left[\begin{array}{ccc|c}-1-\lambda_2&1&0&0\\1&2-\lambda_2&1&0\\0&3&-1-\lambda_2&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}-4&1&0&0\\1&-1&1&0\\0&3&-4&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}-4&1&0&0\\0&3&-4&0\\0&0&0&0\end{array}\right]$$

$$\begin{bmatrix}-4&1&0\\0&3&-4\\0&0&0\end{bmatrix}\begin{bmatrix}x_{21}\\x_{22}\\x_{23}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Rightarrow\begin{cases}-4x_{21}+x_{22}=0\\3x_{21}-4x_{23}=0\end{cases}$$

$$choose\quad x_{21}=1\quad\therefore\quad x_{22}=4,\quad x_{23}=3,\quad x_2=\begin{bmatrix}1\\4\\3\end{bmatrix}$$

check :

$$\begin{bmatrix}-1&1&0\\1&2&1\\0&3&-1\end{bmatrix}\begin{bmatrix}1\\4\\3\end{bmatrix}=\begin{bmatrix}3\\12\\9\end{bmatrix}=3\cdot\begin{bmatrix}1\\4\\3\end{bmatrix}$$

Eigenvector for $\lambda_3=-2$:

$(A-\lambda_3 I)x_3=0$

$$\left[\begin{array}{ccc|c}-1-\lambda_3&1&0&0\\1&2-\lambda_3&1&0\\0&3&-1-\lambda_3&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_{31} + x_{32} = 0 \\ 3x_{31} + x_{33} = 0 \end{cases}$$

$$choose \quad x_{31} = 1 \quad \therefore \quad x_{32} = -1, \quad x_{33} = 3, \quad x_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

check :

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} = (-2) \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Problem 4:

$$x' = Ax$$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

The same matrix was given in problem 3. So we had the same eigenvalues and eigenvectors. We can get the results as following:

$$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}$$

And with the initial conditions:

$$x(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & 4 & -1 & -2 \\ -1 & 3 & 3 & -8 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & 4 & -1 & -2 \\ 0 & 1 & 1 & -3 \end{array}\right] \qquad R3 \Leftarrow \frac{R1+R3}{4}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -3 \end{array}\right] \qquad R2 \Leftarrow \frac{R2+R3}{5}, \quad R1 \Leftarrow R1+R3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad R3 \Leftarrow R3-R2$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] = \left[\begin{array}{c} -1 \\ -1 \\ -2 \end{array}\right] \quad \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = -1 \\ c_3 = -2 \end{cases}$$

$$\therefore x(t) = (-1)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + (-1)\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t} + (-2)\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}$$