Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering

EXAM 2 – Solution

Tuesday, October 31

Problem 1 (24 points):

Find the solution, y(t) for the following second-order linear differential equation and initial conditions with a polynomial driving function: (12 points each for finding the particular and homogeneous parts of the solution.)

$y'' + 6y' + 9y = -27t^2 + 27t;$ $y(0) = -3,$ $y'(0) = 0$

Solution:

Particular solution:

The right side function indicates that a good choice for the particular solution is a polynomial of the same degree as the driving function.

Choose

$$y_p(t) = k_2 t^2 + k_1 t + k_0$$

Therefore,

$$y_p'(t) = 2k_2 t + k_1$$

$$y_p''(t) = 2k_2$$

Substitute into the equation,

$$\begin{aligned} y_p'' + 6y_p' + 9y_p &= (2k_2) + 6(2k_2 t + k_1) + 9(k_2 t^2 + k_1 t + k_0) \\ &= 9k_2 t^2 + (12k_2 + 9k_1)t + 2k_2 + 6k_1 + 9k_0 \\ &= -27t^2 + 27t \end{aligned}$$

Equating the coefficients of the same powers of *x* on both sides,

$$\begin{cases} 2k_2 + 6k_1 + 9k_0 = 0 \\ 12k_2 + 9k_1 = 27 \\ 9k_2 = -27 \end{cases} \quad \Rightarrow \quad \begin{cases} k_0 = -4 \\ k_1 = 7 \\ k_2 = -3 \end{cases}$$

Therefore

$$y_p(t) = -3t^2 + 7t - 4$$

$$y_p'(t) = -6t + 7$$

And,

$$y_p(0) = -4, \quad y_p'(0) = 7$$

$$y_h(0) = y(0) - y_p(0) = -3 - (-4) = 1$$

$$y_h'(0) = y'(0) - y_p'(0) = 0 - 7 = -7$$

Homogeneous solution:

The homogeneous equation is given as

$$y_h''(t) + 6y_h'(t) + 9y_h(t) = 0$$

The characteristic equation is then:

$$\lambda^2 + 6\lambda + 9 = 0$$

$$\Rightarrow \quad (\lambda + 3)^2 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = -3$$

The homogeneous solution is thus

$$y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$y_h'(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$$

Initial values give

$$\begin{cases} y_h(0) = c_1 = 1 \\ y_h'(0) = -3c_1 + c_2 = -7 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 1 \\ c_2 = -4 \end{cases}$$

Therefore

$$y_h(t) = e^{-3t} - 4te^{-3t}$$

$$\therefore y(t) = y_h(t) + y_p(t) = (1 - 4t)e^{-3t} - 3t^2 + 7t - 4$$

Problem 2 (36 points):

For the second order linear differential equation with initial conditions and sinusoidal driving function below:

y'' – y' – 12 = –2cos(2t) – 16sin(2t); y(0) = 7, y'(0) = 2

a) (12 points) Solve for the particular solution of y(t) using the "phasor" method.

The driving function is a sum of harmonic functions of angular frequency 2. So it can be represented in the form *Re[(P+jQ)e^{3jt}]*. Here *P* and *Q* are real numbers.

$$-2\cos(3t) - 16\sin(3t) = \mathrm{Re}[(P + jQ)e^{2jt}]$$
$$= \mathrm{Re}[(P + jQ)(\cos(2t) + j\sin(2t))]$$
$$= P\cos(2t) - Q\sin(2t)$$
$$\Rightarrow \quad P = -2, \quad Q = 16, \quad \therefore \hat{f} = -2 + 16j$$

Therefore, the differential equation can be re-written as

$$y_p'' - y_p' - 12y_p = \mathrm{Re}[(-2+16j)e^{2jt}]$$

The form of the equation suggests that the particular solution is also a harmonic function of angular frequency 2. So we can represent the particular solution in the phasor form

$$\hat{y}_p(t) = \hat{Y}_0 e^{2jt}$$
$$\hat{y}_p'(t) = 2j\hat{Y}_0 e^{2jt}, \quad \hat{y}_p'' = -4\hat{Y}_0 e^{2jt}$$

Substituting these into the differential equation,

$$-4\hat{Y}_0 e^{2jt} - 2j\hat{Y}_0 e^{2jt} - 12\hat{Y}_0 e^{2jt} = (-2+16j)e^{2jt}$$
$$\Rightarrow \qquad \hat{Y}_0(-16-2j) = (-2+16j)$$
$$\therefore \hat{Y}_0 = \frac{-2+16j}{-16-2j} = -j\frac{(-2j-16)}{-16-2j} = -j$$

Therefore

$$y_p(t) = \mathrm{Re}[\hat{y}_p(t)] = \mathrm{Re}[\hat{Y}_0 e^{2jt}] = \mathrm{Re}[-je^{2jt}] = \sin(2t)$$

b) (12 points) Solve for the particular solution of y(t) assuming the solution has a form similar to the form of the rigth-hand-side driving function (the "standard" form). (hint: you should get the same result as you did in part a).

The driving function indicates that a good choice for the particular solution is a trigonometric expression. Let

$$y_p = K\cos(2t) + M\sin(2t)$$

Therefore,

$$y_p' = -2K\sin(2t) + 2M\cos(2t)$$
$$y_p'' = -4K\cos(2t) - 4M\sin(2t)$$

Substituting into the equation

$$[-4K\cos(2t) - 4M\sin(2t)] - [-2K\sin(2t) + 2M\cos(2t)]$$
$$-12[K\cos(2t) + M\sin(2t)] = -2\cos(2t) - 16\sin(2t)$$
$$\Rightarrow \begin{cases} -16K - 2M = -2 \\ 2K - 16M = -16 \end{cases} \Rightarrow \begin{cases} K = 0 \\ M = 1 \end{cases}$$

$$\Rightarrow \quad y_p(t) = \sin(2t)$$

Thus, by both the methods, we get the same particular solution.

c) (12 points) Solve for the homogeneous solution of y(t) and write the total solution, y(t), for the equation.

This is the solution of the differential equation obtained by setting the driving term zero.

$$y_h'' - y_h' - 12 y_h = 0$$

The characteristic equation is then,

$$\lambda^2 - \lambda - 12 = 0$$

$$\Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = -3$$

Therefore, the solution has the form,

$$y_h(t) = c_1 e^{4t} + c_2 e^{-3t}$$

and thus the total solution is,

$$y(t) = y_h(t) + y_p(t) = c_1 e^{4t} + c_2 e^{-3t} + \sin(2t)$$

and,

$$y'(t) = 4c_1 e^{4t} - 3c_2 e^{-3t} + 2\cos(2t)$$

Initial values give,

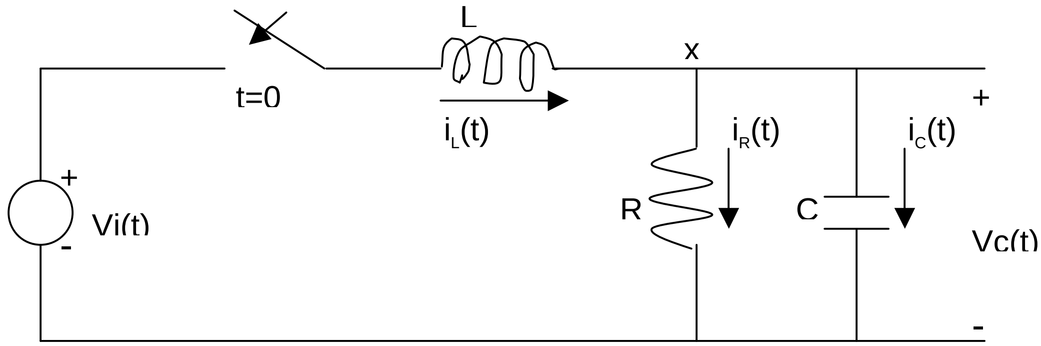
$$\begin{cases} y(0) = c_1 + c_2 = 7 \\ y'(0) = 4c_1 - 3c_2 + 2 = 2 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 3 \\ c_2 = 4 \end{cases}$$

Therefore

$$y(t) = 3e^{4t} + 4e^{-3t} + \sin(2t)$$

Problem 3 (40 points):

Consider the L-R-C circuit below:



For this problem, we are going t solve the voltage Vc(t). The switch is closed at time t=0, and the initial conditions on the circuit are: Vc(0) = 0V; $I_L(0)$ = 0A.

a) (12 points) Write a linear second order ordinary different equation (ODE) for the circuit above. Your answer shuold be in terms of Vc(t) (and its derivatives), R, L, C, and Vi(t).

From the circuit, we have,

$$i_C(t) = CV_C'(t)$$
$$V_L(t) = V_i(t) - V_C(t) = L\,i_L'(t)$$
$$i_R(t)R = V_C(t)$$

Apply KCL at x:

$$i_L(t) - i_R(t) - i_C(t) = 0$$
$$\Rightarrow \quad i_L(t) = i_R(t) + i_C(t) = CV_C'(t) + \frac{V_C(t)}{R}$$
$$Li_L'(t) = L\left[CV_C''(t) + \frac{V_C'(t)}{R}\right] = V_i(t) - V_C(t)$$
$$\Rightarrow \quad V_C''(t) + \frac{V_C'(t)}{RC} + \frac{V_C(t)}{LC} = \frac{V_i(t)}{LC}$$

b) (8 points) Solve the characteristic equation for the above ODE and express your λ's in terms of the circuit components: L, C, and R. What range(s) of R make the system " over-damped" and "under-damped"?

The characteristic equation is,

$$\lambda^2 + \frac{\lambda}{RC} + \frac{1}{LC} = 0$$
$$\Rightarrow \lambda = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{(RC)^2} - \frac{4}{LC}}}{2}; \quad set\ \Delta = \frac{1}{(RC)^2} - \frac{4}{LC}$$

We have,

As $R < \frac{1}{2}\sqrt{\frac{L}{C}},\ \Delta > 0$, over-damped.

As $R > \frac{1}{2}\sqrt{\frac{L}{C}},\ \Delta < 0$, under-damped.

c) (20 points) Let Vi(t) =5V, L=1H, C=0.5F, and R=1ohm. Solve the ODE you derived in a) with these values and the initial conditions given in the problem statement to find a final function for the voltage Vc(t). Is the circuit over-damped, under-damped, or critically-damped?

Now we have,

$$V_C''(t) + \frac{V_C'(t)}{(1)(0.5)} + \frac{V_C(t)}{(1)(0.5)} = \frac{V_i(t)}{(1)(0.5)}$$

$$\Rightarrow \; V_C''(t) + 2V_C'(t) + 2V_C(t) = 10$$

For particular solution, choose

$$V_{Cp}(t) = K_0$$

Plug into the above equation,

$$2K_0 = 10 \quad \Rightarrow \quad K_0 = 5$$

$$\therefore \; V_{Cp}(t) = 5\,(volts) \qquad \Rightarrow V_{Cp}(0) = 5\,(volts)$$

$$and\; V_{Cp}'(t) = 0\,(volts/s) \qquad \Rightarrow V_{Cp}'(0) = 0\,(volts/s)$$

For homogeneous solution, the characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$$

Then we can have the homogeneous solution,

$$V_{Ch}(t) = c_1 e^{(-1+j)t} + c_2 e^{(-1-j)t}$$

$$V_{Ch}'(t) = (-1+j)c_1 e^{(-1+j)t} + (-1-j)c_2 e^{(-1-j)t}$$

Apply the initial values

$$\begin{cases} V_{Ch}(0) = c_1 + c_2 = -5 \\ V_{Ch}'(0) = (-1+j)c_1 + (-1-j)c_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = -\frac{5}{2} + \frac{5}{2}j \\ c_2 = -\frac{5}{2} - \frac{5}{2}j \end{cases}$$

Thus,

$$\begin{aligned} V_{Ch}(t) &= \left(-\frac{5}{2} + \frac{5}{2}j\right)e^{(-1+j)t} + \left(-\frac{5}{2} - \frac{5}{2}j\right)e^{(-1-j)t} \\ &= \frac{5}{2}e^{-t}\left[(-1+j)e^{jt} + (-1-j)e^{-jt}\right] \\ &= \frac{5}{2}e^{-t}\left[-e^{jt} - e^{-jt} + je^{jt} - je^{-jt}\right] \\ &= \frac{5}{2}e^{-t}\left[-2\cos(t) - 2\sin(t)\right] \\ &= 5e^{-t}\left[-\cos(t) - \sin(t)\right] \end{aligned}$$

So,

$$V_C(t) = V_{Ch}(t) + V_{Cp}(t) = 5 + 5e^{-t}\left[-\cos(t) - \sin(t)\right]$$