Carnegie Mellon University

Department of Electrical and Computer Engineering

18-200 **Fall 2000**

Mathematical Foundations of Electrical Engineering
Exam 1 – Solution

Problem 1:

$$Let \quad z = -27j \Rightarrow \begin{cases} r = |z| = 27 \\ \theta = \angle z = \dfrac{3\pi}{2} \end{cases}$$

$$For \sqrt[3]{-27j}:$$

$$\left|\sqrt[3]{z}\right| = \sqrt[3]{r} = 3; \quad \angle\sqrt[3]{z} = \frac{\theta + 2k\pi}{3} \quad k = 0,1,2$$

$$So, \quad \theta_1 = \frac{3\pi}{2\cdot 3} = \frac{\pi}{2}$$

$$\theta_2 = \frac{3\pi}{2\cdot 3} + \frac{2\pi}{3} = \frac{7\pi}{6} \quad (or -\frac{5\pi}{6})$$

$$\theta_2 = \frac{3\pi}{2\cdot 3} + \frac{4\pi}{3} = \frac{11\pi}{6} \quad (or -\frac{\pi}{6})$$

$\therefore$ 3 roots: z_1, z_2, z_3,

$$z_1 = 3e^{j\pi/2} = 3j$$

$$z_2 = 3e^{-j5\pi/6} = 3\left[\cos\left(\frac{5\pi}{6}\right) - j\sin\left(\frac{5\pi}{6}\right)\right] = -\frac{3\sqrt{3}}{2} - j\frac{3}{2}$$

$$z_2 = 3e^{-j\pi/6} = 3\left[\cos\left(\frac{\pi}{6}\right) - j\sin\left(\frac{\pi}{6}\right)\right] = \frac{3\sqrt{3}}{2} - j\frac{3}{2}$$

Problem 2:

$$e^{3z} = -3j$$

$$3z = \ln(-3j) = \ln\left(3e^{j(-\frac{\pi}{2}+2n\pi)}\right) \qquad (or = \ln\left(3e^{j(\frac{3\pi}{2}+2n\pi)}\right))$$

$$= \ln 3 + j(-\tfrac{\pi}{2} + 2n\pi) \qquad (or = \ln 3 + j(\tfrac{3\pi}{2} + 2n\pi))$$

$$= \ln 3 + j\tfrac{(4n-1)\pi}{2} \qquad (or = \ln 3 + j\tfrac{(4n+3)\pi}{2})$$

$$\therefore z = \frac{\ln 3}{3} + j\tfrac{(4n-1)\pi}{6} \qquad (or \ \frac{\ln 3}{3} + j\tfrac{(4n+3)\pi}{6})$$

Problem 3:

$$z = (-3+4j)^{(-2j)}$$

$$\Rightarrow \ln z = (-2j)\ln(-3+4j) = (-2j)\ln\left[5e^{j\tan^{-1}\left(\frac{4}{-3}\right)}\right]$$

$$= (-2j)\left[\ln 5 + j\tan^{-1}\left(\frac{-4}{3}\right)\right]$$

$$= 2\tan^{-1}\left(\frac{-4}{3}\right) - j2\ln 5$$

$$\therefore z = e^{\ln z} = e^{\left[2\tan^{-1}\left(-\frac{4}{3}\right) - j2\ln 5\right]} = e^{2\tan^{-1}\left(-\frac{4}{3}\right)} e^{-j2\ln 5}$$

$$= e^{2\tan^{-1}\left(-\frac{4}{3}\right)}\left[\cos(2\ln 5) - j\sin(2\ln 5)\right]$$

$$= -83.5634 + j6.4709$$

Problem 4:

$$z^2 + (1-j2)z - (2+j4) = 0$$

Use quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$where\ a = 1,\ b = (1-2j),\ c = -(2+4j)$$

$$z = \frac{(-1+j2) \pm \sqrt{(1-j2)^2 + 4(2+j4)}}{2} = \frac{(-1+j2) \pm \sqrt{5+12j}}{2}$$

$$= \frac{(-1+j2) \pm \sqrt{13e^{j\tan^{-1}\left(\frac{12}{5}\right)}}}{2}$$

$$where\ \sqrt{13e^{j\tan^{-1}\left(\frac{12}{5}\right)}} = \sqrt{13}e^{j\tan^{-1}\left[\left(\frac{12}{5}\right)/2\right]}$$

$$= \sqrt{13}\cos\left(\tan^{-1}\left[\left(\tfrac{12}{5}\right)/2\right]\right) + j\sqrt{13}\sin\left(\tan^{-1}\left[\left(\tfrac{12}{5}\right)/2\right]\right)$$

$$= 3 + j2$$

$$\therefore z_1 = \frac{(-1+j2) + 3 + j2}{2} = 1 + j2$$

$$z_2 = \frac{(-1+j2) - (3+j2)}{2} = -2$$

Problem 5:

$$w(z) = \cosh(z) = \frac{e^z + e^{-z}}{2} = \frac{e^x e^{jy} + e^{-x}e^{-jy}}{2}$$

$$= \frac{e^x\left[\cos(y) + j\sin(y)\right] + e^{-x}\left[\cos(y) - j\sin(y)\right]}{2}$$

$$= \frac{e^x + e^{-x}}{2}\cos(y) + j\frac{e^x - e^{-x}}{2}\sin(y)$$

$$= \cosh(x)\cos(y) + j\sinh(x)\sin(y)$$

Before we do derivative, we have to check Cauchy-Riemann condition

$$u_x = \frac{\partial[\cosh(x)\cos(y)]}{\partial x} = \sinh(x)\cos(y); \quad u_y = \frac{\partial[\cosh(x)\cos(y)]}{\partial y} = -\cosh(x)\sin(y)$$

$$v_x = \frac{\partial[\sinh(x)\sin(y)]}{\partial x} = \cosh(x)\sin(y); \quad v_y = \frac{\partial[\sinh(x)\sin(y)]}{\partial y} = \sinh(x)\cos(y)$$

$$\therefore u_x = v_y \quad and \quad u_y = -v_x$$

Now we can take

$$\frac{dw(z)}{dz} = \frac{d(\cosh(z))}{dz} = \sinh(z)$$

Problem 6:

$$f(z) = ze^{(3z-j\pi)} = ze^{3z}e^{-j\pi} = -ze^{3z}$$

$$= -(x+jy)e^{3(x+jy)} = -e^{3x}(x+jy)[\cos(3y) + j\sin(3y)]$$

$$= -e^{3x}\left[(x\cos(3y) - y\sin(3y)) + j(y\cos(3y) + x\sin(3y))\right]$$

a)

$$u(x,y) = -e^{3x}(x\cos(3y) - y\sin(3y))$$

$$v(x,y) = -e^{3x}(y\cos(3y) + x\sin(3y))$$

b)

$$u_x = -\left[e^{3x}\cos(3y) + 3e^{3x}(x\cos(3y) - y\sin(3y))\right]$$

$$= -e^{3x}\cos(3y) - 3e^{3x}x\cos(3y) + 3e^{3x}y\sin(3y)$$

$$u_y = -e^{3x}[-3x\sin(3y) - \sin(3y) - 3y\cos(3y)]$$

$$= 3e^{3x}x\sin(3y) + e^{3x}\sin(3y) + 3e^{3x}y\cos(3y)$$

$$v_x = -\left[e^{3x}\sin(3y) + 3e^{3x}(y\cos(3y) + x\sin(3y))\right]$$

$$= -e^{3x}\sin(3y) - 3e^{3x}y\cos(3y) - 3e^{3x}x\sin(3y)$$

$$v_y = -e^{3x}[\cos(3y) - 3y\sin(3y) + 3x\cos(3y)]$$

$$= -e^{3x}\cos(3y) + 3e^{3x}y\sin(3y) - 3e^{3x}x\cos(3y)$$

$$\therefore u_x = v_y \quad and \quad u_y = -v_x$$

f(z) is analytic.

c)

$$f(z) = -ze^{3z}$$

$$i) \quad f(z = j\pi/2) = -j\frac{\pi}{2}e^{j\frac{3\pi}{2}} = -j\frac{\pi}{2}(-j) = -\frac{\pi}{2}$$

$$ii) \quad f'(z) = -e^{3z} - 3ze^{3z} = -e^{3z}(3z+1)$$

$$\therefore f'(z = j\pi/2) = -e^{j\frac{3\pi}{2}}(j\tfrac{3\pi}{2} + 1) = j(j\tfrac{3\pi}{2} + 1) = -\tfrac{3\pi}{2} + j$$