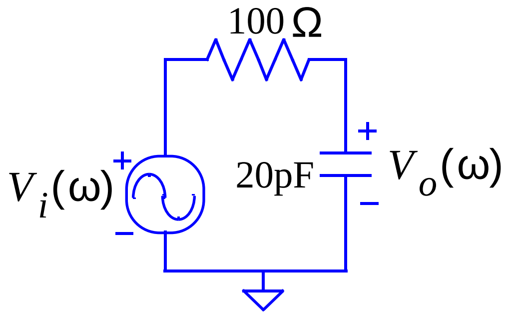
Frequency Response

- Sometimes we will design analog circuits to attenuate certain frequencies while amplifying others --- Filters and decoupling circuits
- In all cases there will be some finite bandwidth due to the nonidealities associated with the transistors and other components
- The output signal phase will also be shifted differently (relative to the input signal) as a function of frequency
- For analog design we generally view plots of the magnitude and phase as a function of frequency, ω, radians/sec to understand these behaviors

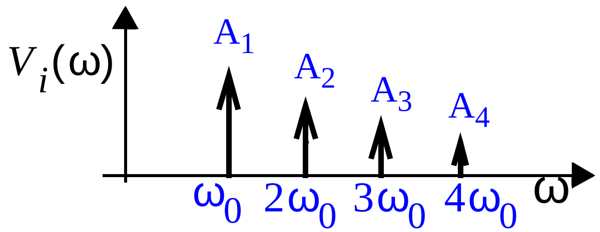
- If we consider only frequency responses to design the circuit, how do we know what is happening in terms of the transient response? And do we care?

Simple RC Circuit Example

- For analog design we generally view plots of the magnitude and phase as a function of frequency, ω, radians/sec

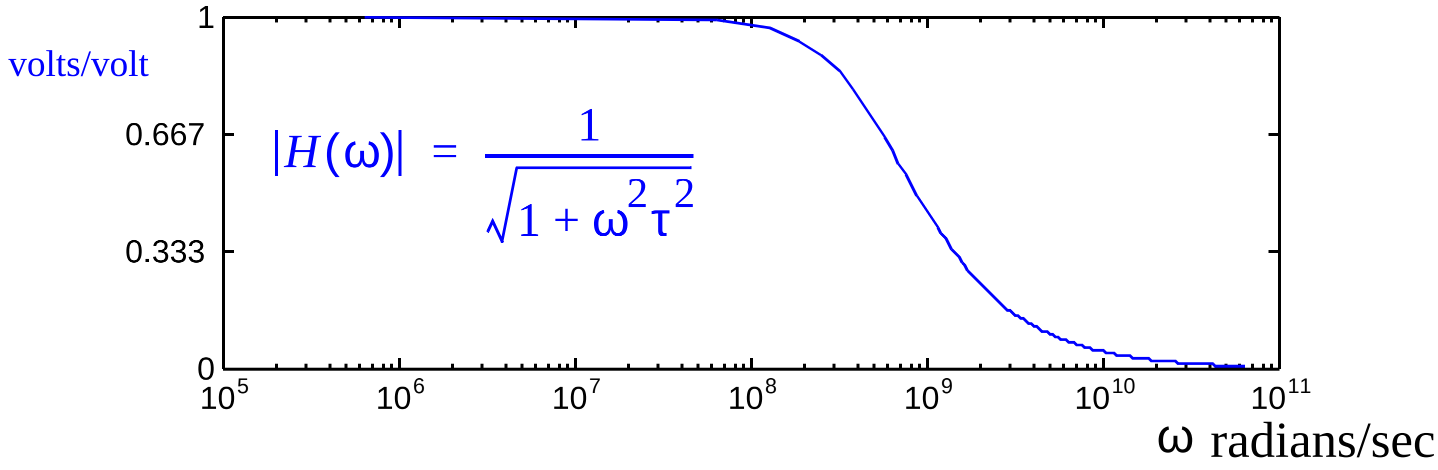


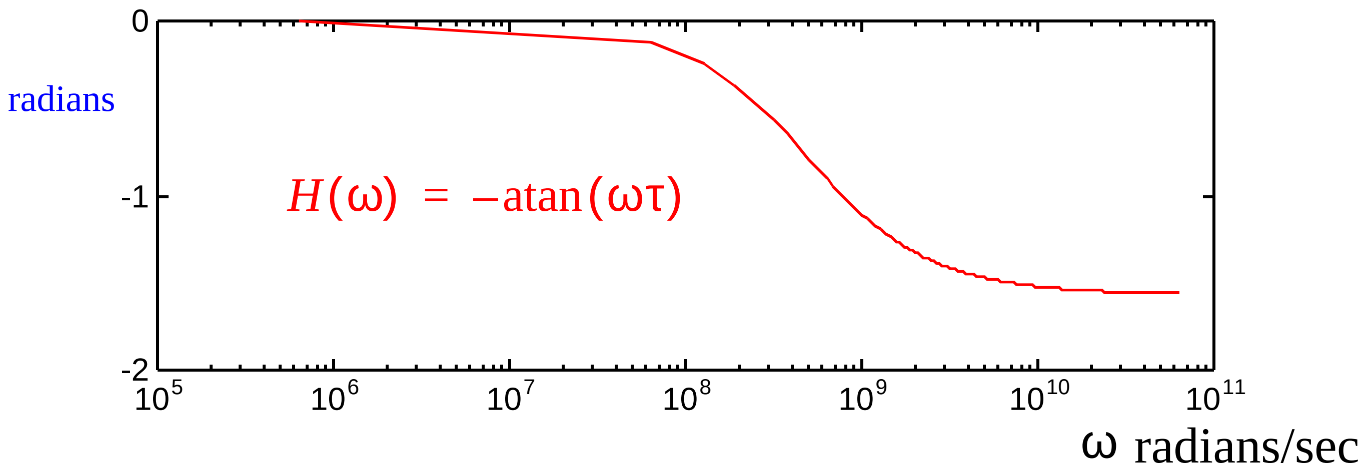
$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)}$$



$$V_i(t) = a_{avg} + \sum_{n=1}^{\infty} A_n \cos(n\omega_o t - \theta_n)$$

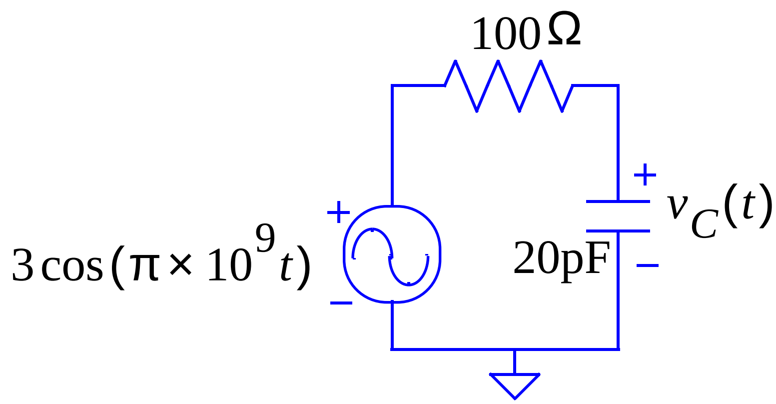
Transfer Function Magnitude and Phase



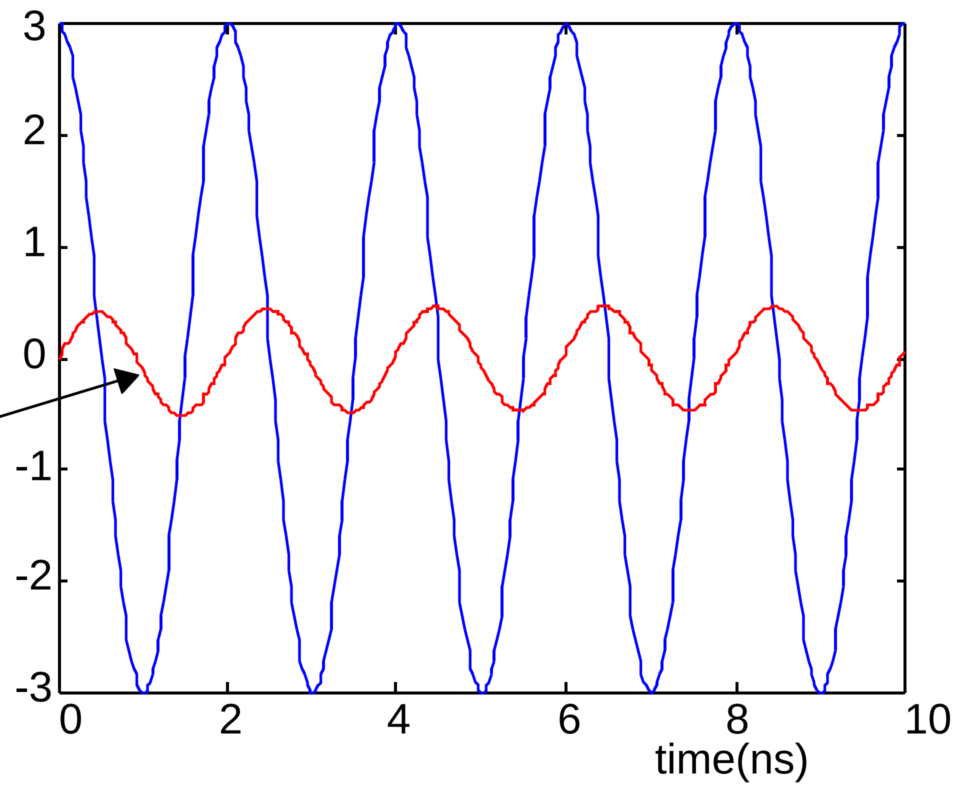


- The steady state response of a cosine input signal is modified in terms of phase and magnitude as displayed on the plots

Steady State Response



$v_{in} = 3\cos(\pi \times 10^{9} t)$

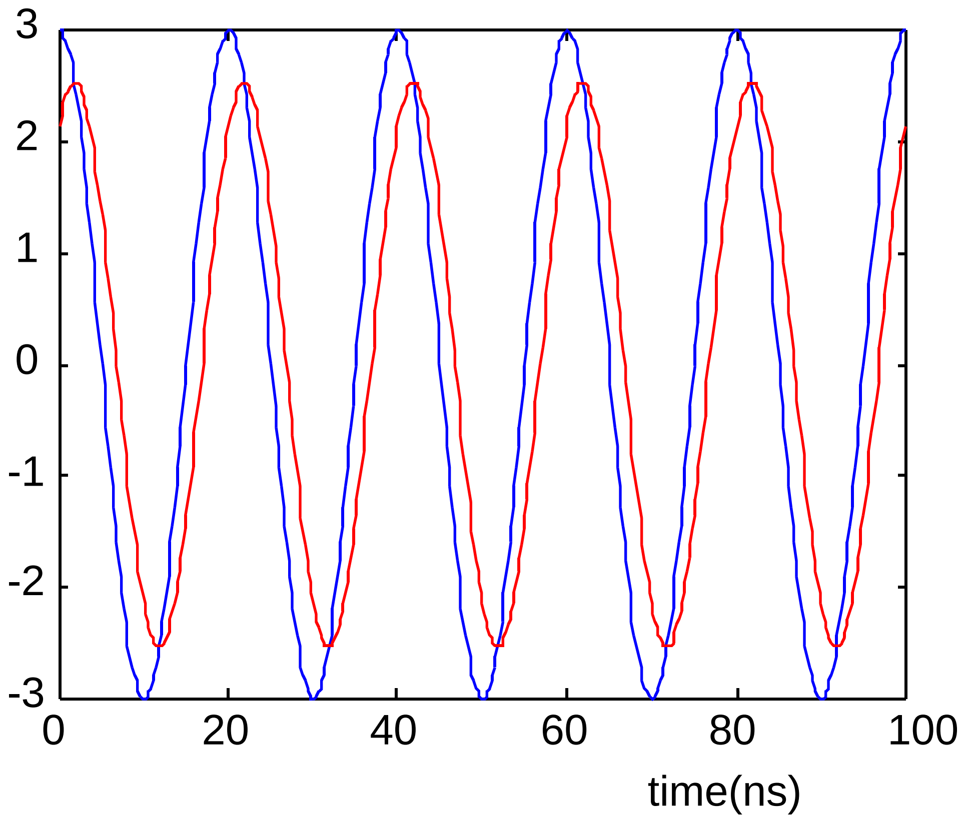


steady state response

Change in SS Response as Frequency is Varied

- Note that lower frequency signals have less attenuation of magnitude and less phase shift, as can be seen from the frequency domain plots

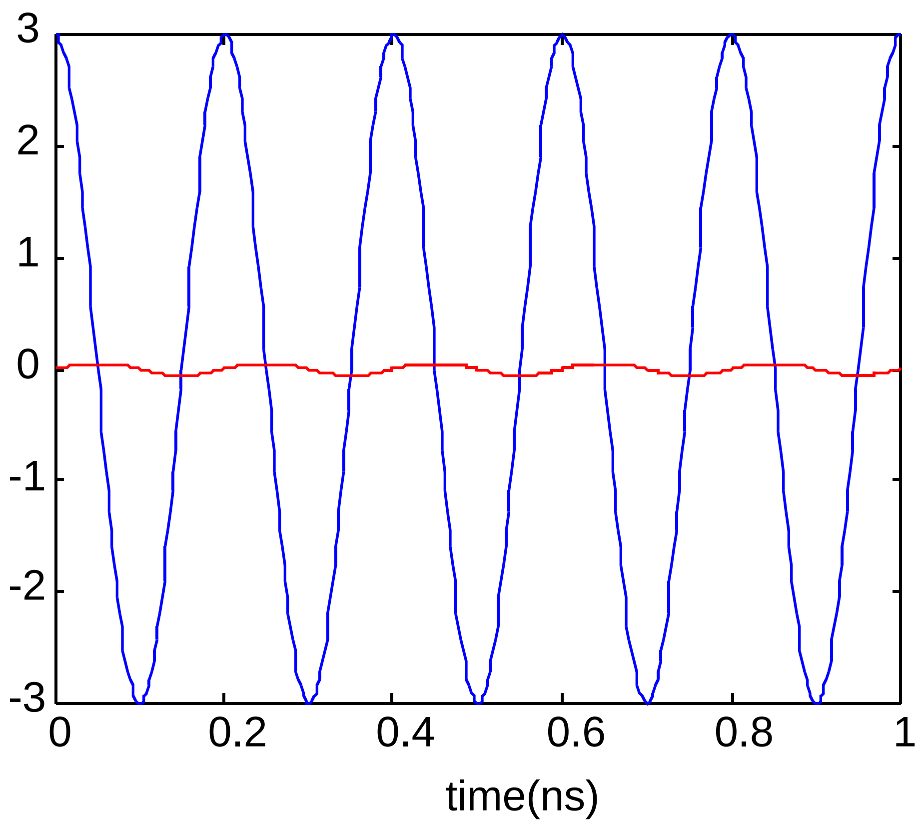
$$v_{in} = 3\cos(\pi \times 10^8 t)$$



High Frequency Response

- For frequencies significantly beyond the breakpoint in the magnitude plot, the response will start to vanish

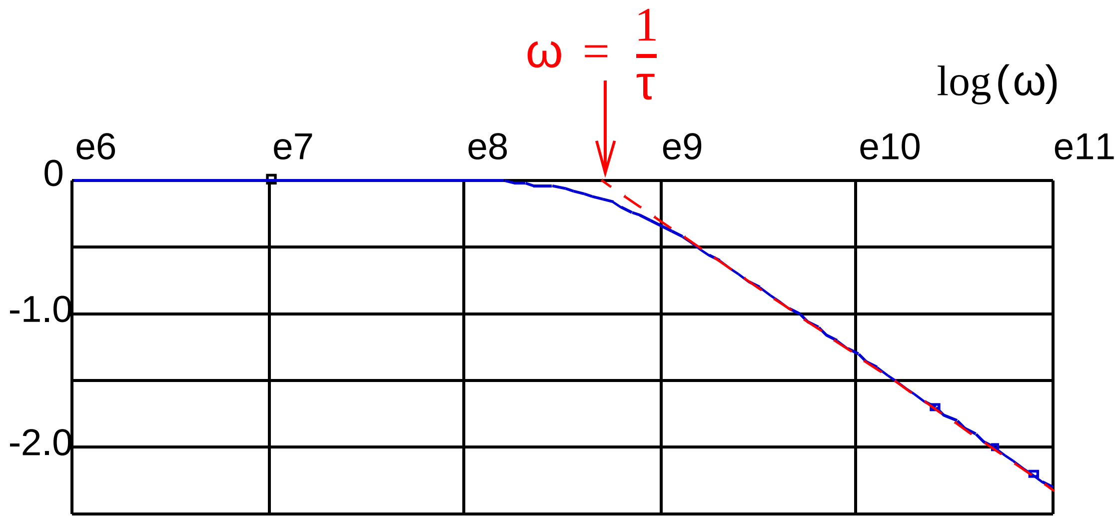
$$v_{in} = 3\cos(\pi \times 10^{10} t)$$



Magnitude on a log Scale

- The breakpoints in these frequency domain characteristics are related to the natural frequencies of the circuit
- Plotting the magnitude on a log-log scale we can see this relationship
- For our RC example, the time constant, τ, is 2ns, therefore the natural frequency is 5e8.

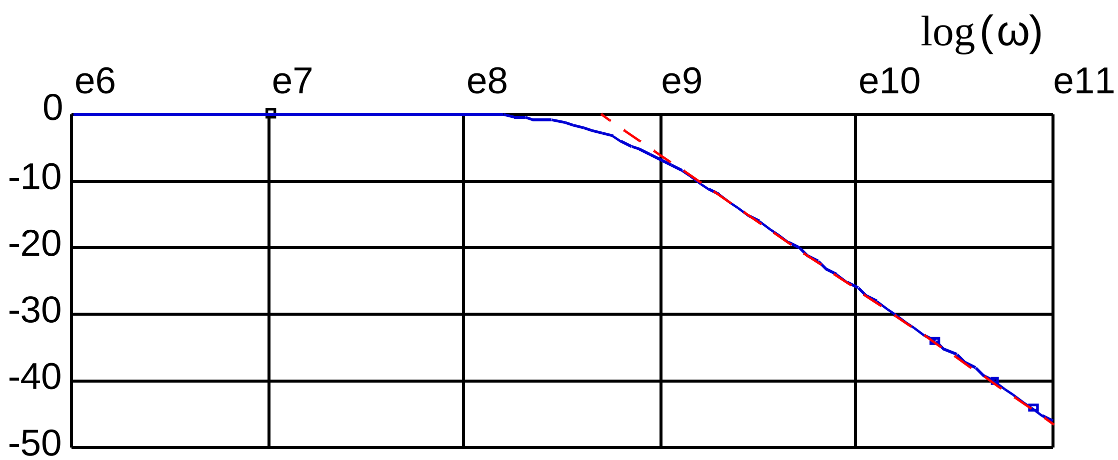
$$\log\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right)$$



Magnitude on a decibel (dB) Scale

- Magnitudes are generally plotted on a dB scale:

$$20\log\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right)$$



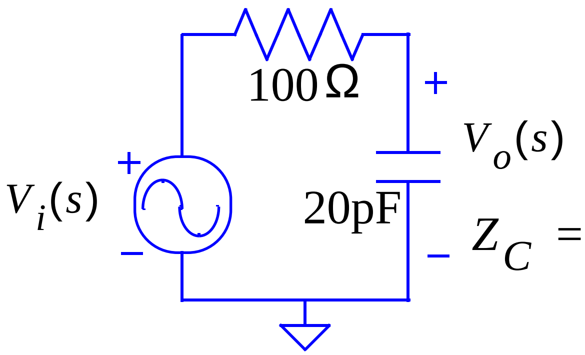
- Magnitude (for this single time constant example) falls off at asymptotic rate of 20dB/decade, or 6dB/octave (an octave is a 2x change in frequency)
- This relation to circuit natural frequencies also holds for higher order circuits
- Allows us to quickly estimate (visualize) the frequency response based on the natural frequencies

Natural Frequencies

- You may not have seen Laplace Transforms yet, but like phasors they represent a transformation to the (complex) frequency domain that makes it easy to solve for natural frequencies, s's, or reciprocal time constants, τ 's

$$s \leftrightarrow j\omega$$

$$Z_R = R$$



$$Z_C = \frac{1}{sC}$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

Poles

- With Laplace transform terminology there is a pole at -1/RC for this RC circuit
- A pole represents a value for s for which $H(s)$ is infinite.

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC} = \frac{\frac{1}{RC}}{\frac{1}{RC} + s}$$

- However, the transfer function is *not* infinite at the real frequency, ω

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 + j\omega RC}$$

Poles and Natural Frequencies

- It is important to note that natural frequencies and time constants have positive magnitude, while poles are negative (negative real parts for RLC)

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC} = \frac{1}{1 + \frac{s}{p}}$$

- p is equal to 1/RC and can be thought of as the natural frequency
- The pole which makes H(s) infinite, however, is s=-p
- We know that if we solved for the time domain response, that the s term in the assumed solution form would have to be a negative value:

$$Ae^{st}$$

Bode Plot

- Once we know the magnitude of the pole, p, for this transfer function, we can use straight line estimates (on the log-log scale) to approximate the frequency response plot

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + \frac{j\omega}{p}}$$

$$|H(\omega)| = \frac{|1|}{\left|1 + \frac{j\omega}{p}\right|} \qquad 20\log(|1|) - 20\log\left(\left|1 + \frac{j\omega}{p}\right|\right) \qquad \text{in dB}$$

- We calculate the asymptotes for the magnitude of this function

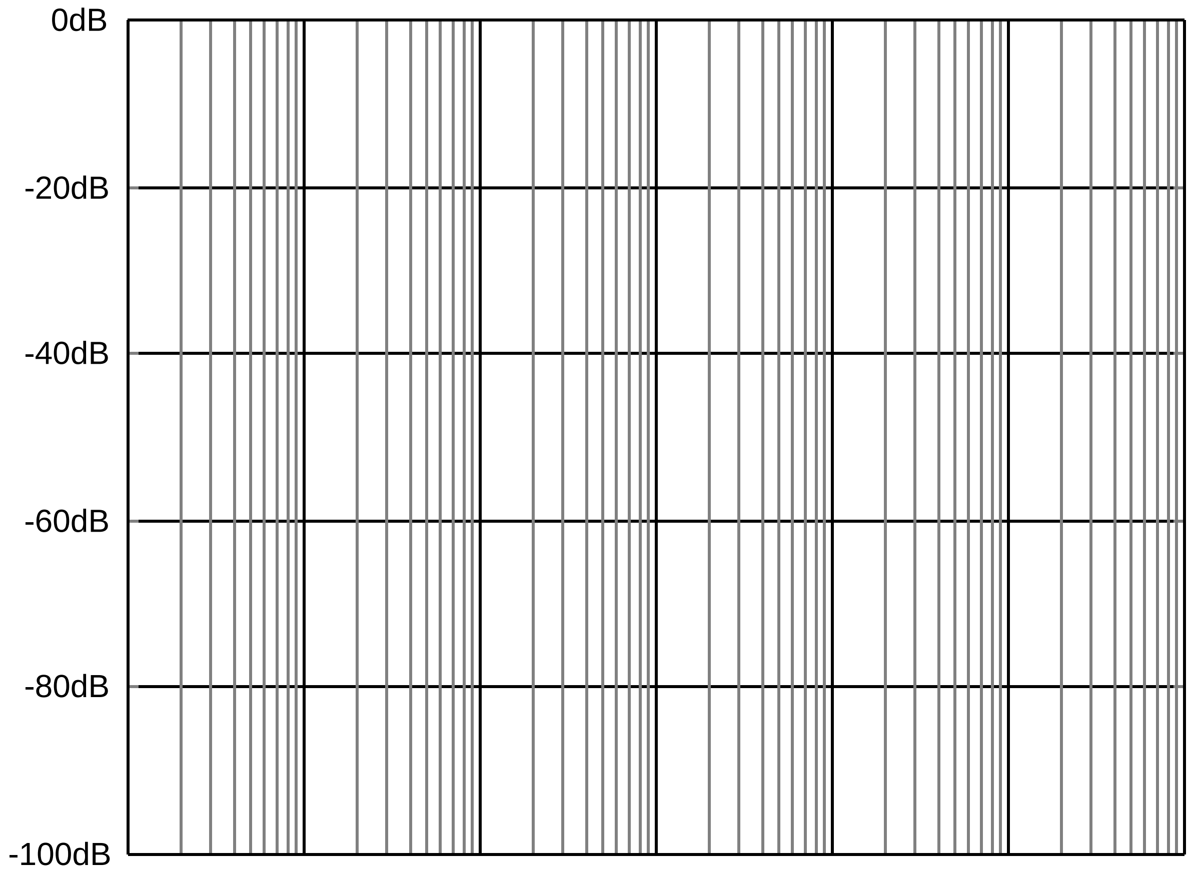
For $\omega \ll p$ $\quad |H(\omega)| \quad 0dB$

For $\omega \gg p$ $\quad |H(\omega)| \quad -20\log\left(\frac{\omega}{p}\right) = -[20\log(\omega) - 20\log(p)](dB)$

- Where do these asymptotes intersect?

Bode Plot

- With frequency plotted on a log scale, we can quickly sketch the asymptotes
- The maximum error at the breakpoint in the curve is known to be 3dB



Bode Plot: Phase

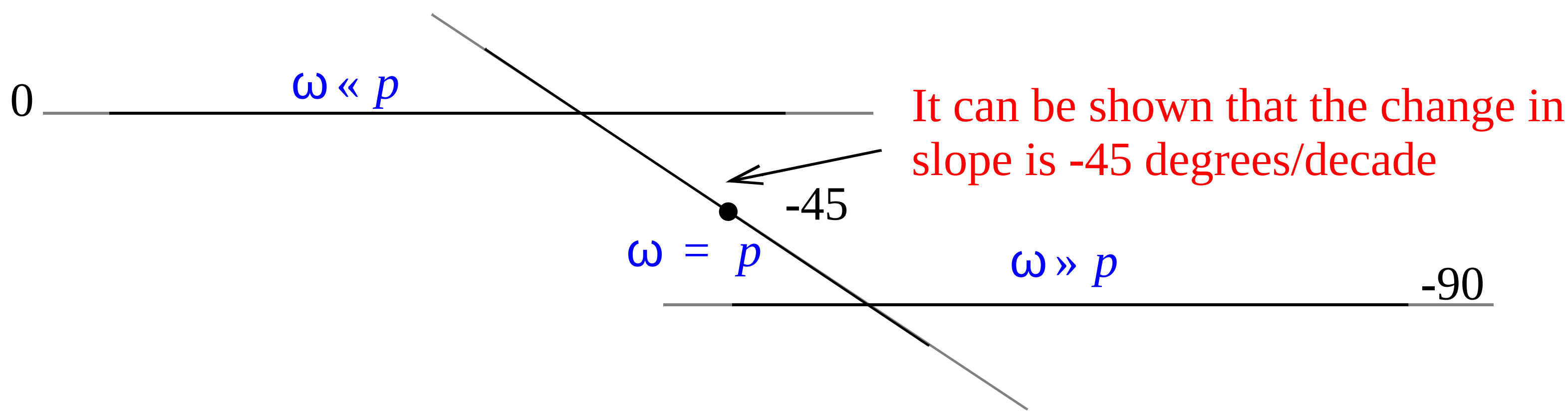
- The phase plot (for this single pole) can be sketched in a similar way

$$H(\omega) = \frac{1}{1 + \frac{j\omega}{p}} \qquad H(\omega) = -\mathrm{atan}\left(\frac{\omega}{p}\right)$$

For $\omega \ll p$ $\quad H(\omega) \quad 0(radians)$ $\quad$ 0 degrees

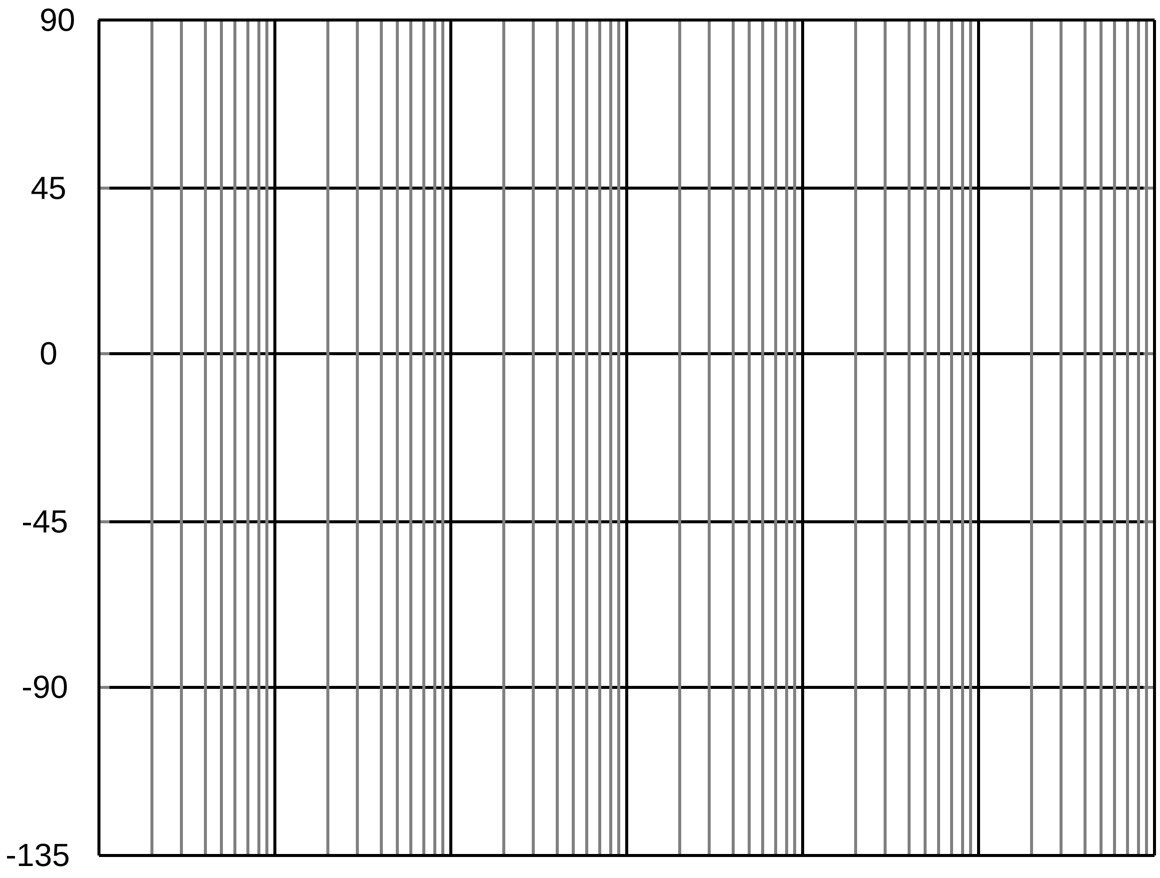
For $\omega \gg p$ $\quad H(\omega) \quad -\frac{\pi}{4}(radians)$ $\quad$ -90 degrees

For $\omega = p$ $\quad H(\omega) \quad -\frac{\pi}{8}(radians)$ $\quad$ -45 degrees



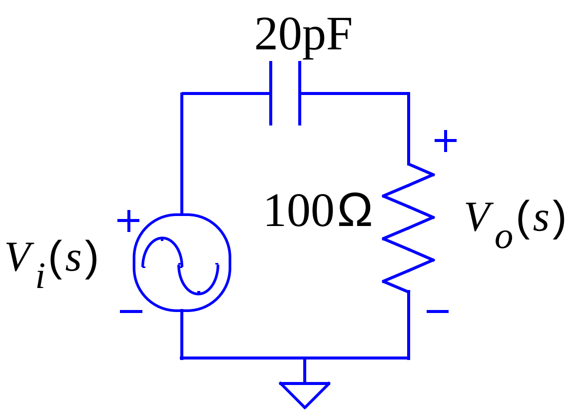
Phase Plot

- Maximum error at the breakpoint is 5.7 degrees



Poles and Zeros

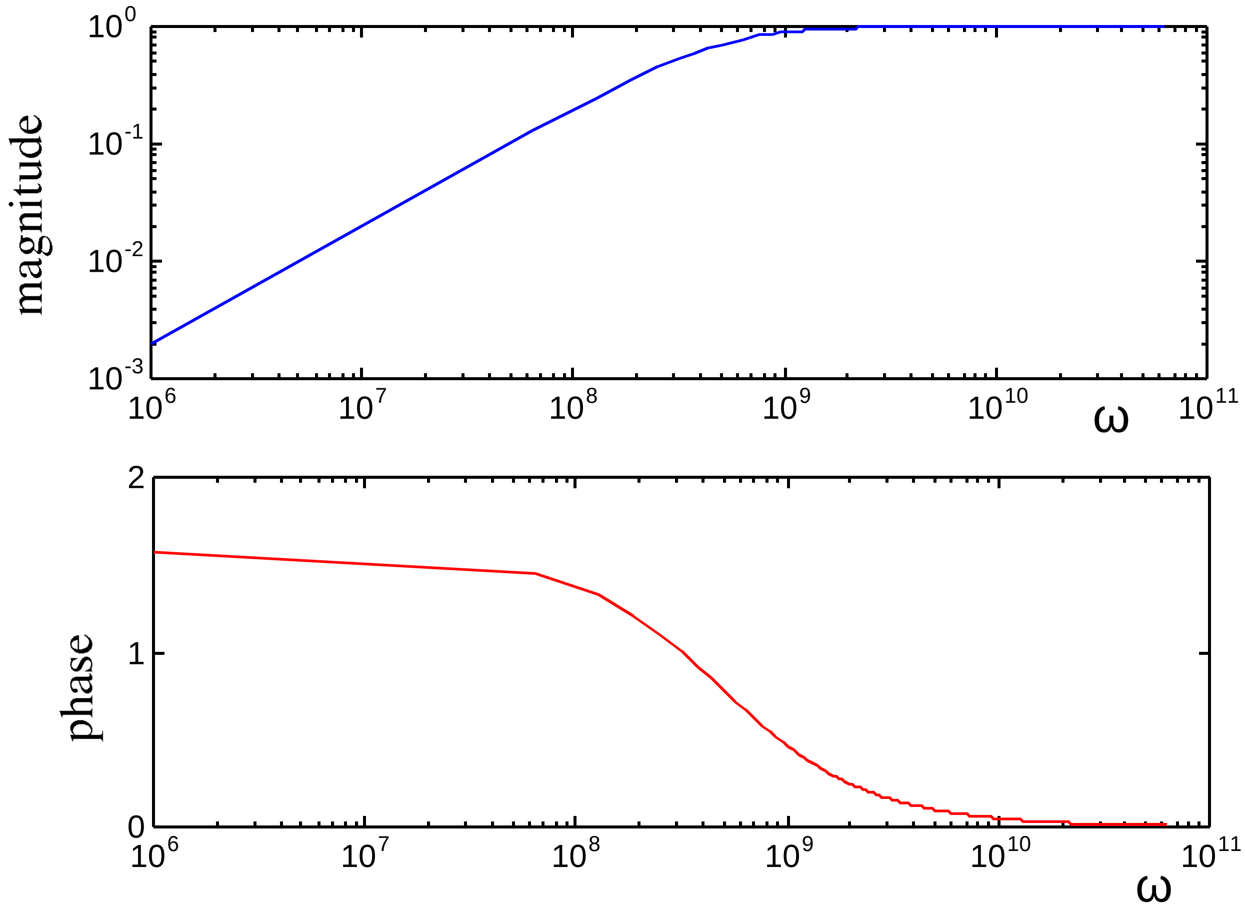
- We'll solve circuits in terms of s, just like the book
- But we'll only consider sinusoidal steady state problems, therefore, $s \rightarrow j\omega$
- We'll also use the terminology of pole to refer to the natural frequency magnitude
- A related term is a zero
- Example of a circuit with a zero at s=0 --- Actually a transmission zero at $\omega = 0$ in this case:



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{\frac{1}{sC} + R} = \frac{sRC}{1 + sRC}$$

Magnitude on a decibel (dB) Scale

- Swapping the R and C changes a low-pass filter into a high-pass filter



- The values of the zeros and poles signify the breakpoints and direction

Bode Plot

- Once we know the pole and zero values we can apply a Bode approximation

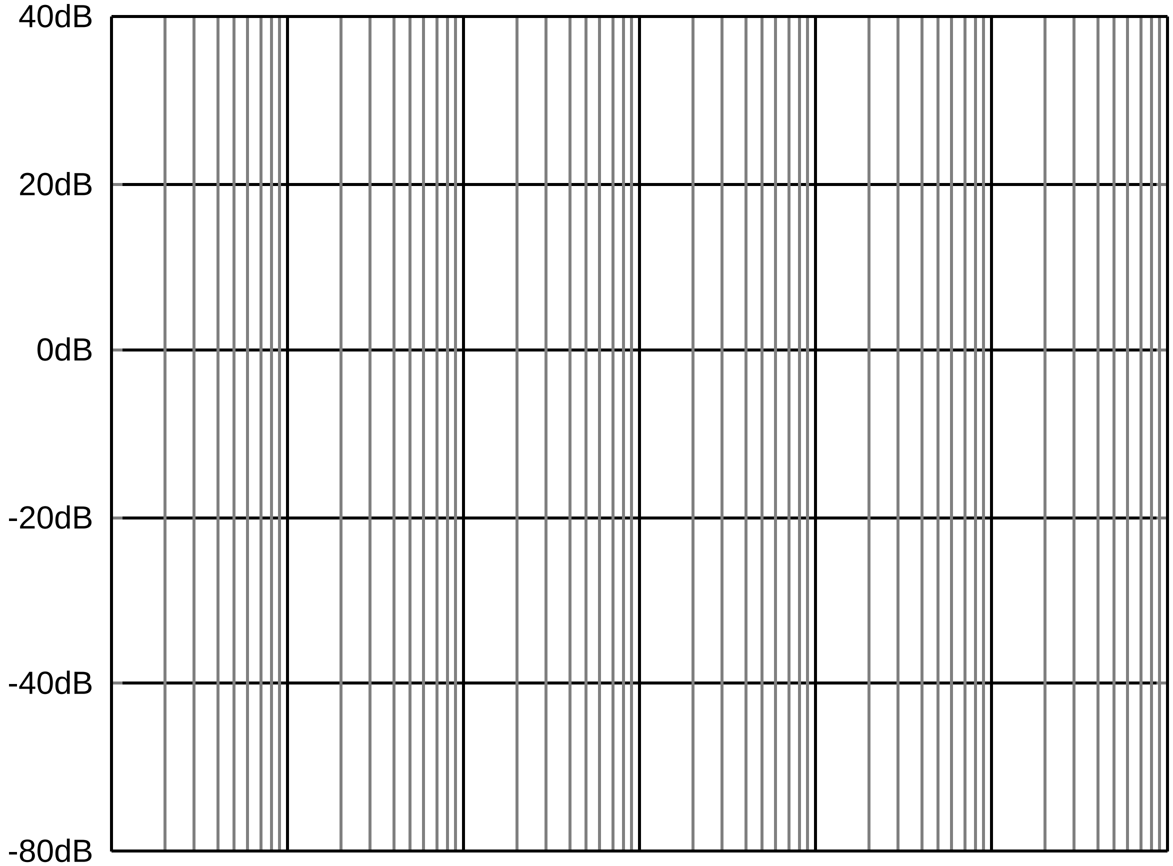
$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\frac{\omega}{p}}{1 + \frac{j\omega}{p}}$$

$$|H(\omega)| = \frac{\left|j\frac{\omega}{p}\right|}{\left|1 + \frac{j\omega}{p}\right|} \qquad 20\log\left(\left|j\frac{\omega}{p}\right|\right) - 20\log\left(\left|1 + \frac{j\omega}{p}\right|\right)$$

- Pole term is the same as before
- Zero term is 0dB at breakpoint, and increasing at a rate of 20dB/decade otherwise
- Note that zeros create asymptotes that are increasing with frequency, while poles create asymptotes that are decreasing with frequency
- We add all of the asymptotes together to get the overall response

Bode Plot

- For each term in the transfer function expression:
  1. Find the direction and slope of the asymptote
  2. Find one point through which the asymptote passes



Poles and Zeros of Larger Circuits

- Bode plots can be used for higher-order circuits too
- But higher order circuits will have more poles and zeros, and transfer functions of the form:

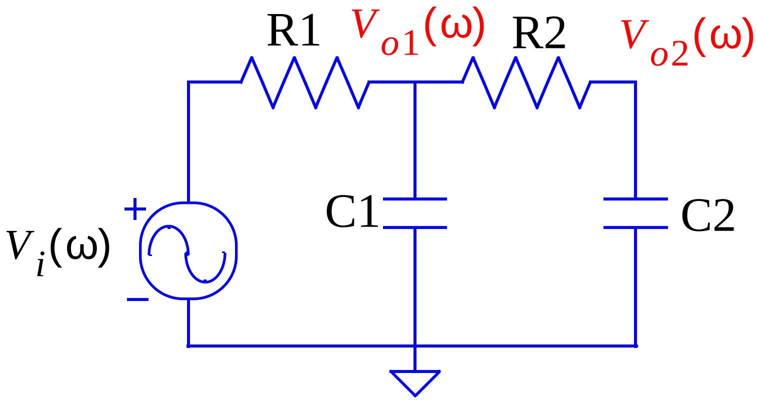
$$H(\omega) = K\frac{\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots\left(1+\frac{j\omega}{z_m}\right)}{\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\cdots\left(1+\frac{j\omega}{p_n}\right)}$$

- We would expect that there will always be more finite poles than zeros, why?

- What does the K-term represent?

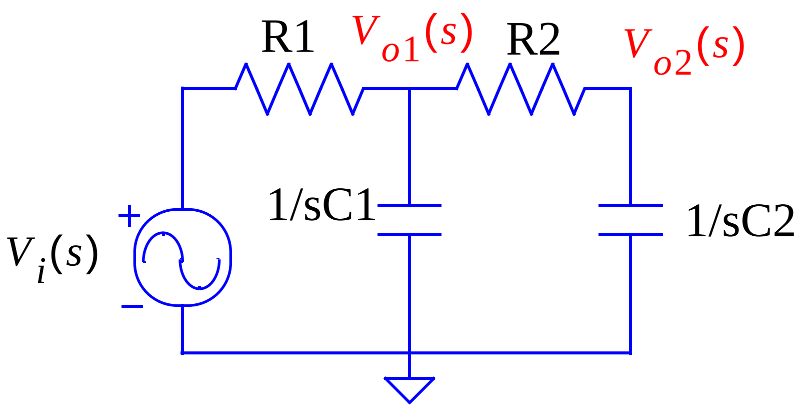
2nd Order Example

- The following RC circuit will have 2 poles and 1 zero for the transfer function from the input to node 1 or 2:



$$H_i(\omega) = K\frac{\left(1 + \frac{j\omega}{z_1}\right)}{\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)}$$

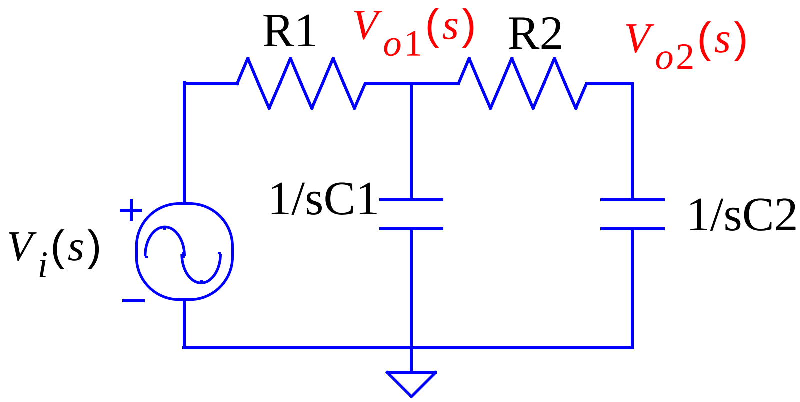
- You may know how to solve for the poles (natural frequencies) of a circuit by formulating the differential equations and using an assumed solution of: Ae^{st}
- An easier way is to use the following circuit to solve for the poles and zeros:



$$H_i(s) = \frac{V_o(s)}{V_i(s)}$$

2nd Order Example

- Write the nodal equations just like you would for phasors

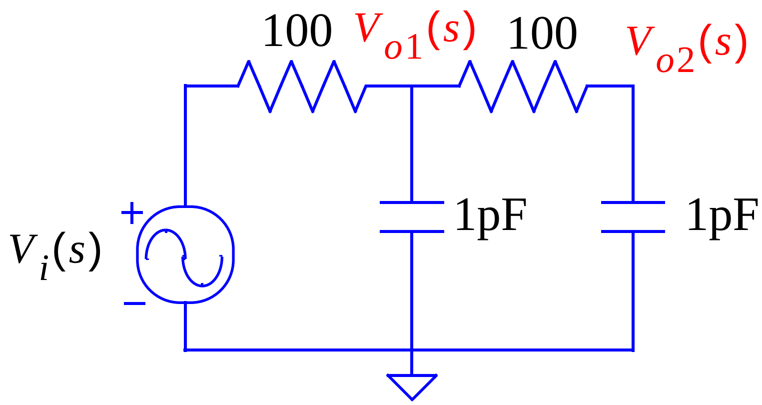


- Transfer functions to nodes 1 and 2 are:

$$H_1(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{R_1R_2C_1C_2}(1 + sR_2C_2)}{s^2 + s\left(\frac{1}{R_2C_2} + \frac{1}{R_2C_1} + \frac{1}{R_1C_1}\right) + \frac{1}{R_1R_2C_1C_2}}$$

$$H_2(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{R_1R_2C_1C_2}}{s^2 + s\left(\frac{1}{R_2C_2} + \frac{1}{R_2C_1} + \frac{1}{R_1C_1}\right) + \frac{1}{R_1R_2C_1C_2}}$$

Numerical Example



$$H_1(s) = \frac{\dfrac{(1+10^{-10}s)}{10^{-20}}}{s^2 + s(0.03\times 10^{12}) + 10^{20}} = 10^{20}\,\frac{\left(1+\dfrac{s}{10^{10}}\right)}{s^2 + s(0.03\times 10^{12}) + 10^{20}}$$

- What is the dc gain?

- What is the zero? Does it represent a transmission zero?

Numerical Example

- Rearrange the terms to recognize the poles

$$H_1(s) = 10^{20} \frac{\left(1 + \frac{s}{z}\right)}{s^2 + s(0.03 \times 10^{12}) + 10^{20}} = 10^{20} \frac{\left(1 + \frac{s}{z}\right)}{(s + p_1)(s + p_2)}$$

  - Where: $p_{1,2} = -(-1.5 \times 10^{10} \pm 1.118 \times 10^{10})$

$$H_1(s) = \frac{10^{20}}{p_1 p_2} \frac{\left(1 + \frac{s}{z}\right)}{\left(\frac{s}{p_1} + 1\right)\left(\frac{s}{p_2} + 1\right)}$$

1.0

- So we can write the sinusoidal steady state transfer function as:

$$H_1(j\omega) = \frac{\left(1 + \frac{j\omega}{z}\right)}{\left(\frac{j\omega}{p_1} + 1\right)\left(\frac{j\omega}{p_2} + 1\right)}$$

Numerical Example

- Transfer function can be expressed as the product of the pole and zero terms:

$$H_1(j\omega) = \frac{\left(1 + \frac{j\omega}{z}\right)}{\left(\frac{j\omega}{p_1} + 1\right)\left(\frac{j\omega}{p_2} + 1\right)} = \left(1 + \frac{j\omega}{z}\right)\frac{1}{\left(\frac{j\omega}{p_1} + 1\right)}\frac{1}{\left(\frac{j\omega}{p_2} + 1\right)}$$

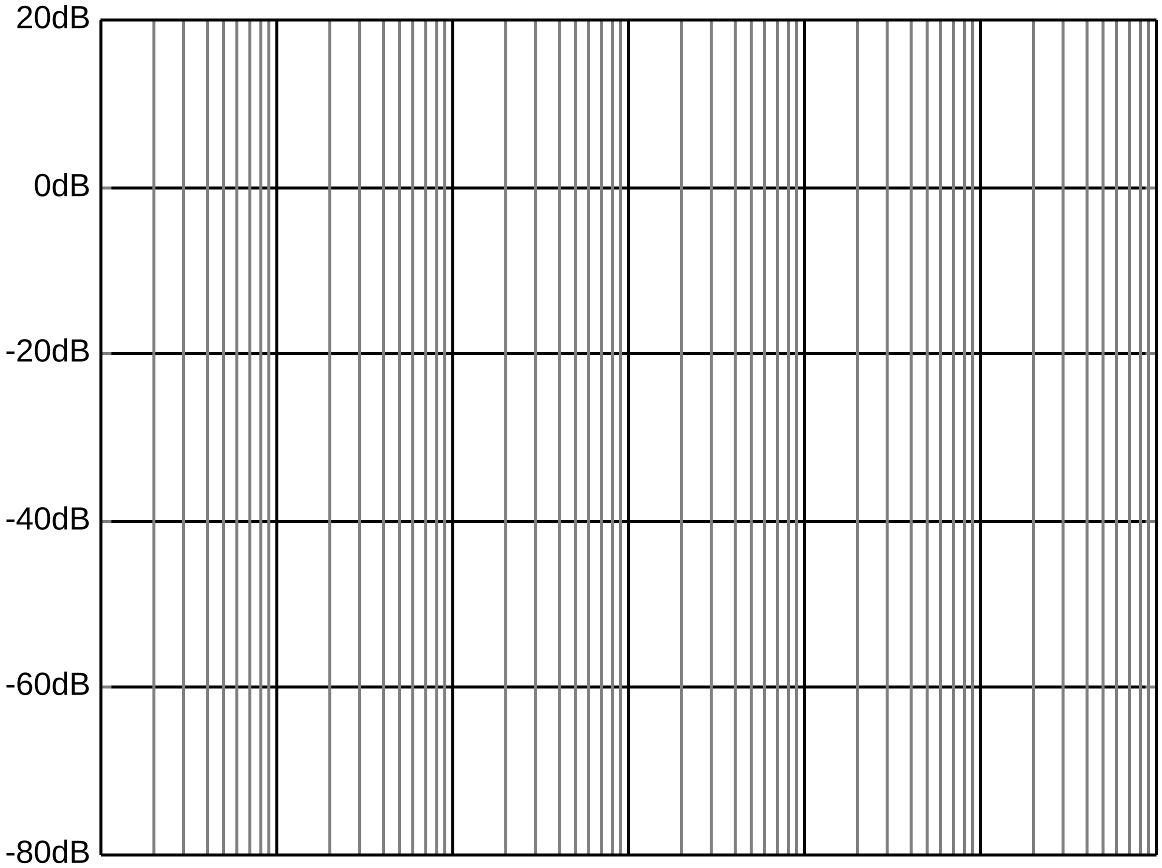
- If we measure the magnitude in dB, then all of the terms can be separated:

$$\left|H_1(j\omega)\right|_{dB} = 20\log\left[\left(1 + \frac{j\omega}{z}\right)\frac{1}{\left(\frac{j\omega}{p_1} + 1\right)}\frac{1}{\left(\frac{j\omega}{p_2} + 1\right)}\right] =$$

$$20\log\left[\left(1 + \frac{j\omega}{z}\right)\right] + 20\log\left[\frac{1}{\left(\frac{j\omega}{p_1} + 1\right)}\right] + 20\log\left[\frac{1}{\left(\frac{j\omega}{p_2} + 1\right)}\right]$$

Bode Plot

- Add the asymptotes for each of the individual pole and zero contributions



Phase Numerical Example

- Starting again with the transfer function in product form:

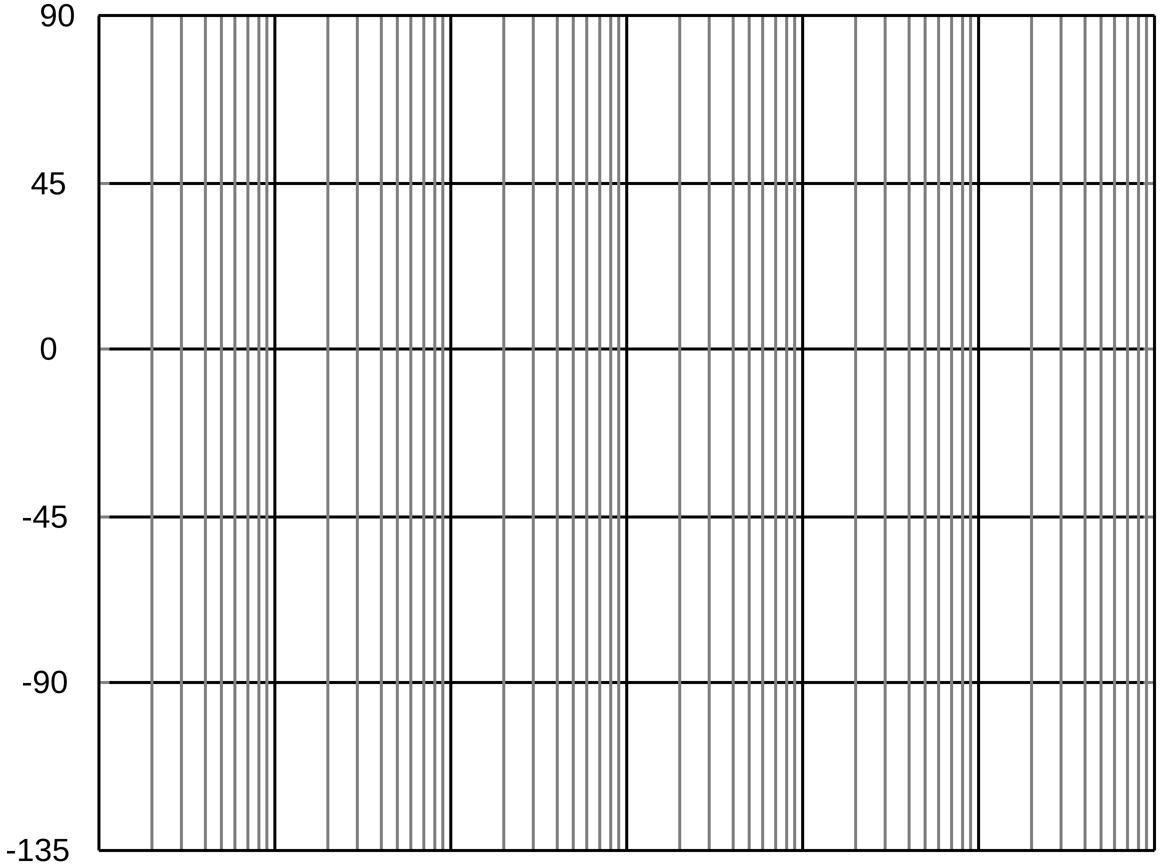
$$H_1(j\omega) = \left(1 + \frac{j\omega}{z}\right) \frac{1}{\left(\frac{j\omega}{p_1} + 1\right)} \frac{1}{\left(\frac{j\omega}{p_2} + 1\right)}$$

- Each term represents a complex number which can be expressed in polar coordinate form

$$H_1(j\omega) = H_z \quad \mathrm{atan}\left(\frac{\omega}{z}\right) \; H_{p1} \quad -\mathrm{atan}\left(\frac{\omega}{p_1}\right) \; H_{p1} \quad -\mathrm{atan}\left(\frac{\omega}{p_2}\right)$$

- From which it is apparent that all of the phase angle terms add
- So we can add the phase-shift asymptotes too

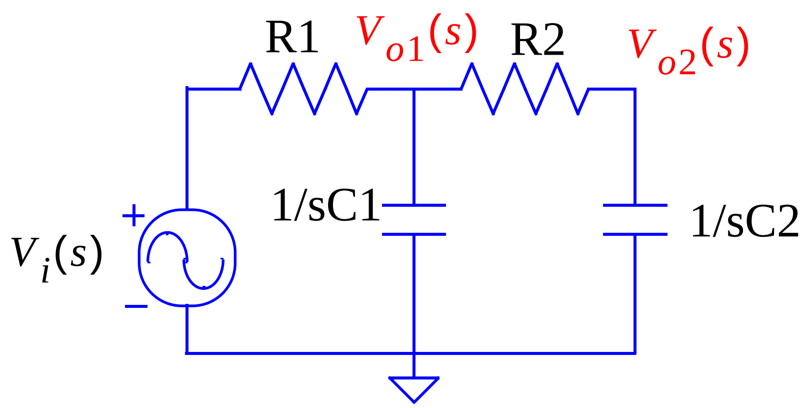
Phase Plot



Zeros at Node 2

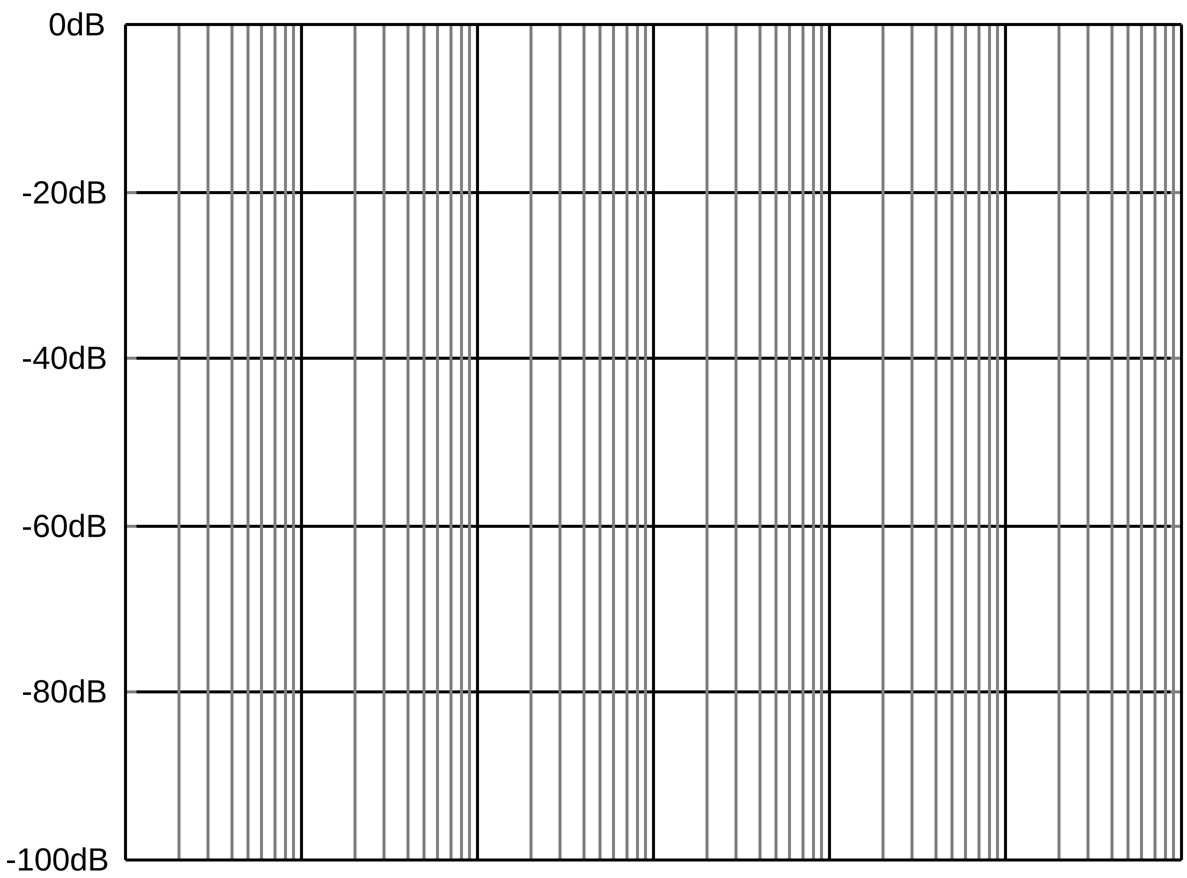
$$H_2(j\omega) = \frac{1}{\left(\frac{j\omega}{p_1} + 1\right)\left(\frac{j\omega}{p_2} + 1\right)}$$

- The zero at node 1 is $1/R_2C_2$, but the response at node 2 does not have any finite zeros --- does this make sense?



Bode Plot

- The transfer functions share the same two poles, their responses are different due to the zero



Phase Plot