Partially Ordered Set

A partially ordered set \((S, \leq)\) is a set \(S\) combined with a binary relation \(\leq\) indicating that one element comes before another. \(\leq\) must satisfy:

1) Reflexive: \(a \leq a\)
2) Antisymmetric: if \(a \leq b\) and \(b \leq a\) then \(a = b\)
3) Transitive: if \(a \leq b\) and \(b \leq c\), then \(a \leq c\)

* For a pair \((a, b)\), if \(a \leq b\) or \(b \leq a\), then \(a \& b\) are comparable.

If all pairs are comparable, we have a totally ordered set. Otherwise, partially ordered.

Example: Power Set \(T = \{u, v, w, \emptyset\}\)

\[ S = \mathcal{P}(T) = \{\emptyset, \{u\}, \{v\}, \{w\}, \{u, v\}, \{u, w\}, \{v, w\}, \{u, v, w\}\} \]

\(a, b \in S\), \(a \leq b\) iff \(a \subseteq b\).

Hasse Diagram

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Ex: Let \(S = \{a, b, c\}\) \(a \leq b, b \leq c, c \leq a\)
\(a \neq b \neq c\)
Is this a poset? No - transitivity is violated.
Complete Lattices

A complete lattice is a poset where all subsets have a supremum (join) and an infimum (meet).

Supremum = smallest upper bound
Infimum = greatest lower bound

We write \((S, \leq, \wedge, \vee, \bot, \top)\)

\(\bot = \emptyset\)

\(\top = S\)

Example: Think back to power set poset. \((P(T), \subseteq)\)

\(\exists u \in S \lor \exists w \in S = \exists u, w \in S\)

\(\forall \exists u \in S, \exists w \in S = \exists u, w \in S\)

Ex Define a poset that is not a complete lattice.

[Diagram of a poset with labeled elements and arrows]
key Question: How do we decide to deny/allow access?

Define both policies and requests as arrays of sets.

Request: \( T^g = \left[ \{IP, Name\}, \{Ad, Data\}, \{Ads\}, \{E\} \right] \)

Policy Clause: \( T^c = \left[ \{E\}, \{3\}, \{3\}, \{3\}, \{3\}, \{3\} \right] \)

**Allow** Define partial order \( E \) over vectors:

\( T^g \preceq T^c \) iff \( \forall \) attributes \( x \), \( T^g \cdot x \preceq T^c \cdot x \) where

\( E_x \) is defined as: \( T_x \preceq T'_x \) iff \( \forall \) \( v \in T_x \), \( \exists \ v' \in T'_x : v \preceq v' \)

**Allow** if \( T^g \preceq T^c \).

**Deny** Define \( \Pi \) as follows: Pointwise, where

\( T_x \Pi T'_x = \bigvee_{v \in T_x} v \wedge \bigvee_{v' \in T'_x} v' \subseteq \bigvee_{v \in T_x} v \wedge \bigvee_{v' \in T'_x} v' \subseteq \bigvee_{v \in T_x} v \wedge \bigvee_{v' \in T'_x} v' \)

i.e. find largest lower bound \( \forall \) pairs from 2 sets.

**Deny**

\( T^g = \left[ \{IP\}, \{T\} \right] \) \( T^c = \left[ \{Name\}, \{T\} \right] \)

\( T^g \Pi T^c = \left[ \{IP\}, \{T\} \right] \)

Is there at least 1 \( \perp \) in \( T^g \Pi T^c \)?

Yes / No

**Allow** / **Deny**