Introduction to Elliptic Curve Cryptography

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Elliptic Curve Cryptography

- Public Key Cryptosystem

- Duality between Elliptic Curve Cryptography and Discrete Log Based Cryptography
  - Groups / Number Theory Basis
  - Additive group based on curves

- What is the point?
  - Less efficient attacks exist so we can use smaller keys than discrete log / RSA based cryptography

Computing Dlog in $(Z_p)^*$ (n-bit prime p)

Best known algorithm (GNFS):

run time exp($\tilde{O}(\sqrt[3]{n})$)

cipher key size	modulus size	Elliptic Curve group size
80 bits	1024 bits	160 bits
128 bits	3072 bits	256 bits
256 bits (AES)	**15360** bits	512 bits

As a result: slow transition away from (mod p) to elliptic curves

Discrete Logs

- Let $p = 2q + 1$ where p, q are large primes
- $\mathbb{Z}_p$ is the group of integers modulo p
- $\left|\mathbb{Z}_p\right| = 2q$
- $G_q = QR\left(\mathbb{Z}_p\right)$ is the quadratic residue subgroup of $\mathbb{Z}_p$
- $\left|QR\left(\mathbb{Z}_p\right)\right| = q$, subgroup of prime order
- Every element $g \in G_q$ is a generator, pick a random one
- Pick secret x, compute $g^x mod\ p$
- Public: (p, q, g, g^x) Secret: x

- Discrete Log Assumption: Given **Public** it is hard to find **Secret**

Outline

- Elliptic curves over reals
- Elliptic curves over Z_p
- ECDH and ECDSA

Elliptic Curves

- Consider the following equation:
$$y^2 = x^3 + ax + b$$
- Idea: we pick (a, b) and form a group which is a set containing all of the points that satisfy the equation

- This group will be defined with a very special addition operation which introduces an additional imaginary point

Example

Not all curves are valid elliptic curves

- Left: $y^2 = x^3$ has a “cusp”
- Right: $y^2 = x^3 - 3x + 2$ has a “self intersection”

- In general we require: $4a^3 + 27b^2 \neq 0$

- Observation: curves are symmetric about the point $y = 0$

Elliptic Curves as a Group

- Groups are sets defined over some operation with some structure / properties

- $G = \{(x, y)\colon\ y^2 = x^3 + ax + b\}$

- Define an operation denoted by '+' such that:
  - If $p_1, p_2 \in G,\ p_1 + p_2 \in G$ (Closure)
  - $(p_1 + p_2) + p_3 = p_1 + (p_2 + p_3)$ (Associative)
  - $\exists 0\ \ s.t.\ \forall p\ \ p + 0 = 0 + p = p$ (Identity)
  - $\forall p\ \exists p^{-1}\ s.t.\ p + p^{-1} = 0$ (Inverse)

– Curves will form an abelian group
  - $p_1 + p_2 = p_2 + p_1$ (Communitive)

The Group Operation

- **Not typical point-wise addition!**
- What is this 0 element?
  - $y^2 = x^3 + ax + b$ does not include (0, 0) if $b \neq 0$
- How do we know inverses exist if we don't know what the 0 element is?
- How do we maintain closure?
  - $(x, y) + (x, y) = (2x, 2y)$ for typical pointwise addition which in general does not lie on the curve

The Group Operation

- Let $P, Q, R \in G$, such that a line passes through all of them, then group operation is:

$$P + Q + R = 0$$

- This is strange, we have a relationship between points that lie along but no clear notion of traditional addition

- We can use the relationship to define a more traditional form of addition:

$$P + Q = -R$$

The Group Operation

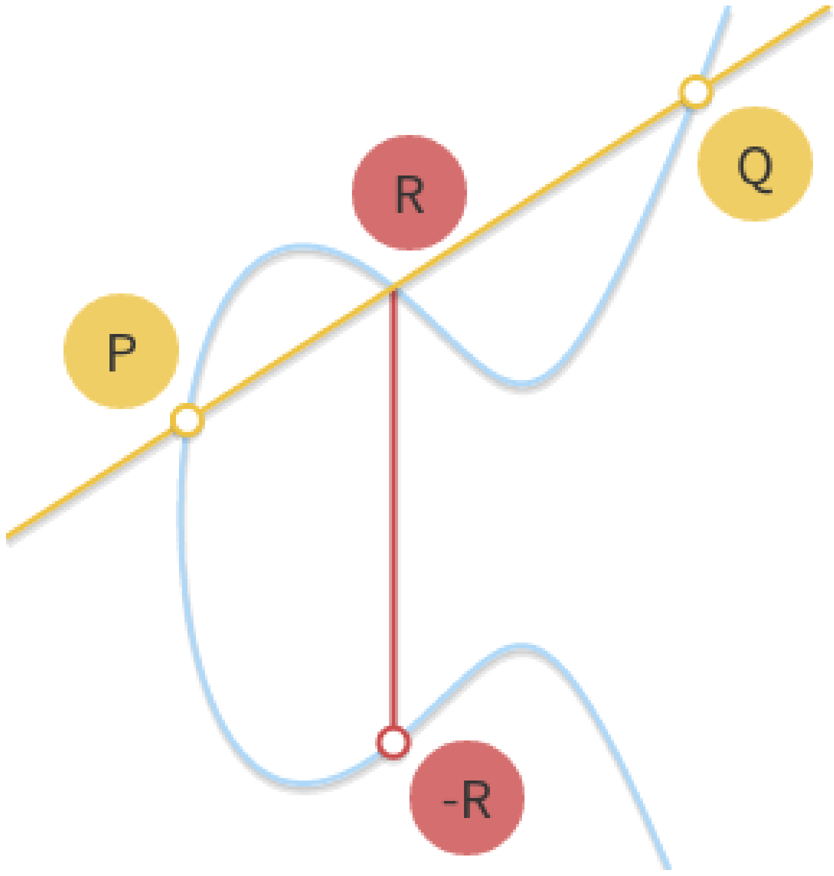
- $P + Q = -R$

- $R = (x_r, y_r), \quad -R = (x_r, -y_r)$
- What happens if we want to compute $-R + R$?
  - What third point on the curve lies on the line defined by $(R, -R)$?

- We say this is the point defined at infinity, we denote it by 0, and it is the additive identity

- $-R + R = 0$

- Adjust our definition of the group:
- $G = \{(x, y)\colon\ y^2 = x^3 + ax + b\} \cup \{0\}$



The Group Operation (Geometric)

- Given $G = \{(x, y)\colon\ y^2 = x^3 + ax + b\} \cup \{0\}$, calculate $P + Q$
  - Geometrically, figure out the third point R such that a line goes through P, Q, R and then set $\mathrm{P} + Q = -R$

- What could possibly go wrong?
  - P or Q could be 0
    - 0 Is the identity under the group operation, so $P + 0 = 0 + P = P$
  - P = -Q
    - This is the case of $-R + R = 0$ which was defined by the vertical line
  - P = Q
    - Imagine tangent to P, use that to find R. $P + P = -R$ describes the line tangent to P that intersects at R
  - There is no 3rd point
    - This occurs when the line is tangent to exactly one of P or Q. Suppose the line is tangent to P, then from before we have $P + P = -Q$ which gives us $P + Q = -P$
    - If line is tangent to Q, then $Q + Q = -P$ which would give us $P + Q = -Q$

Algebraic Solution

- Let $P \neq Q$, line defined by P, Q has slope

$$m = \frac{y_P - y_Q}{x_P - x_Q}$$

- Intersection with point $R = (\mathrm{x_R}, \mathrm{y_R})$:
  - $x_R = m^2 - x_P - x_Q$
  - $y_R = y_P + m(x_R - x_P) = \mathrm{y_Q} + \mathrm{m}(\mathrm{x_R} - \mathrm{x_Q})$

- How would we check that this is correct?
  - Check if $(x_R, y_R) \in G$, if it is then correct with high probability

Multiplication

- We have defined addition, so now we can define multiplication
- $n * P = P + P + \ \dots + P\ (n - times)$
- Inefficient for multiplying by large numbers
- Use doubling algorithm, analogue of repeated squaring algorithm for exponentiation
- Calculate 19 (6 Additions):
  - A = 1+1 = 2
  - B = A + A = 2 + 2 = 4
  - C = B + B = 4 + 4 = 8
  - D = C + C = 16
  - 19 = D + A + 1

Back to Discrete Logs

- In the discrete log setting, exponentiation was easy, but logs were hard
  - g^x – Easy, $\log_g g^x$ – Hard

- In the elliptic curve setting, multiplication is easy but division is hard
  - We still call division "logarithm" even though its really division here

- We used the asymmetry of these operations in the discrete log setting to do key exchange / encryption, can do a similar thing with elliptic curves

Fields

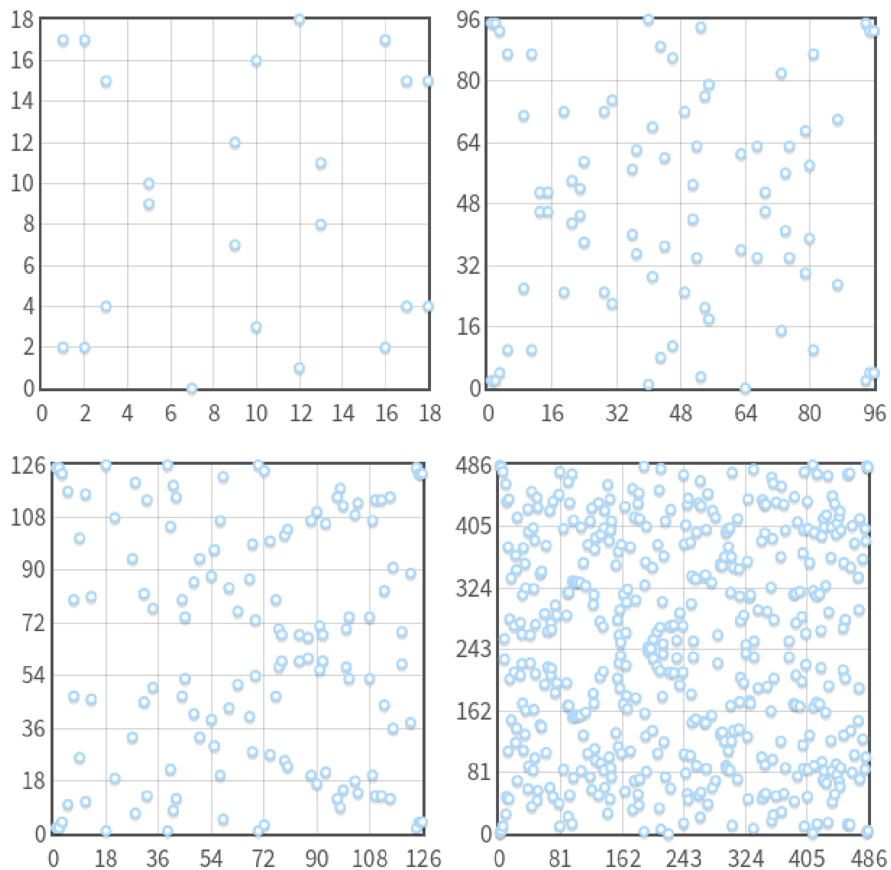
- A field is a set $\mathbb{F}$ with two operations $(+, \times)$ that has the following properties:
  - $\mathbb{F}$ is an abelian group under $+$
  - The non-zero elements of $\mathbb{F}$ are an abelian group under $\times$
  - $a(b+c) = ab + ac \ \ \forall a, b, c \in \mathbb{F}$ (Distributive)

Elliptic Curves Over a Field

- Note: $\mathbb{Z}_n^*(+,\times)$ is a field when n is prime
- Refine the definition of the curve group again:
- $G = \left\{ \begin{matrix} (x, y) \in (\mathbb{F}_P)^2\colon\ y^2 = x^3 + ax + b \ (mod\ p) \\ \wedge\, 4a^3 + 27b^2 \neq 0 \ (mod\ p) \end{matrix} \right\} \cup \{0\}$

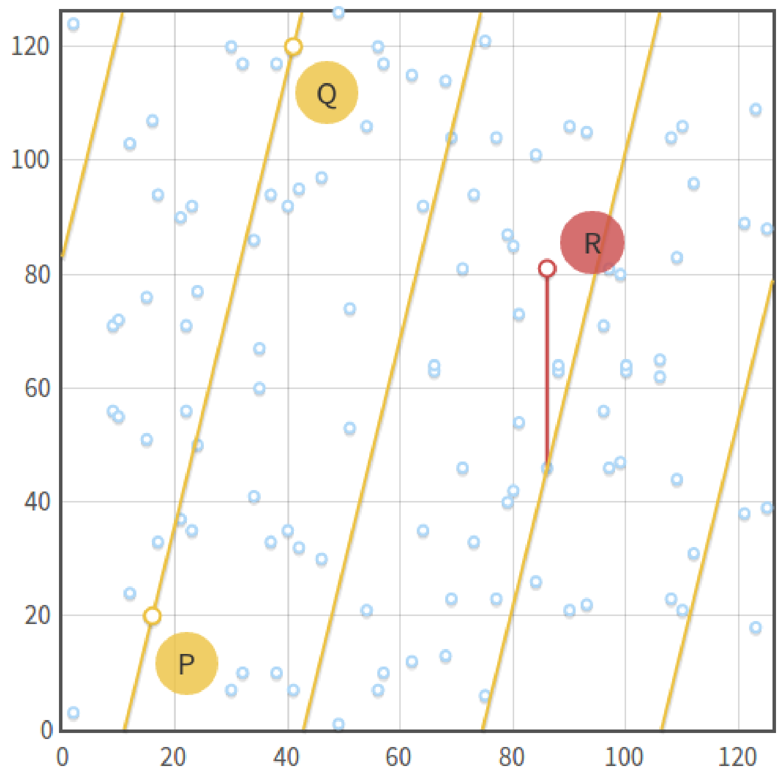
- Curves are now defined only at discrete points and not over the smooth lines that we had before

Elliptic Curves Over a Field



$y^2 = x^3 - 7x + 10\ (mod\ p)$ where $p = 19, 97, 127, 487$

Operation for Curves Over a Field



Curve $y^2 = x^3 - x + 3\ (mod\ 127)$, $P = (16, 20)$, $Q = (41, 120)$

Operation for Curves Over a Field

- The addition operation that we defined before works exactly the same on curves defined over a field

- All of the special cases are handled exactly the same as before

- Intersection with point $R = (\mathrm{x_R}, \mathrm{y_R})$ still computed as:
  - $x_R = m^2 - x_P - x_Q mod\ p$
  - $y_R = y_P + m(x_R - x_P) mod\ p = \mathrm{y_Q} + \mathrm{m}\left(\mathrm{x_R} - \mathrm{x_Q}\right) mod\ p$

Order of Elliptic Curve Group

- # of unique points in the group
  - Could simply try and count them, but there are too many for this to be possible

- Efficient algorithms for computing this exist

Subgroups of Elliptic Curve Groups

- In the discrete log setting, we selected a generator g and computed $\{g^0, g^1, \ldots\}\ mod\ p$

- This group generated by the generator had an order that divided the order of the parent group by Lagrange's Theorem

- In Elliptic Curves we can select a point P which is like a generator and compute $\{0P, P, 2P, 3P, \ldots\}\ mod\ p$, we call this a **Base Point**

- This operation will also generate a cyclic subgroup of the Elliptic curve group whose order divides the order of the parent group

Subgroups of Elliptic Curve Groups

- Suppose we pick a point, P, how can we find the order of the subgroup generated by P?

- Let N be the order of the parent group
- Let $N = p_1^{k_1} p_2^{k_2}$ ... be the prime factorization of N
- Let n be the order of the subgroup

- Idea: take all divisors of N, given by the prime factorization, and sort them smallest to largest, call them n. The order of the subgroup is the smallest n such that nP = 0.

Finding Base Point With High Order

- We will want to find a base point that generates a subgroup with prime order that is as high as possible

- Let $h = \frac{N}{n}$ we will call h the **cofactor** of the subgroup

- Let n be the largest prime factor in the prime factorization of N

- $NP = 0$ because N is an integer multiple of any point P
- $n(hP) = 0$ by re-writing $N = nh$
- This tells us that the point $hP = G$ has order n unless $G = 0$
- G is a generator of a cyclic subgroup of prime order n

ECDH – Elliptic Curve Diffie-Hellman

- Regular Diffie-Hellman:
  - Alice has secret a and computes g^a
  - Bob has secret b and computes g^b
  - They exchange and compute g^{ab}
  - Key insight: it is hard for an adversary to compute g^{ab} from g^a, g^b

- ECDH Setting, Public Parameters: (p, a, b, G, n, h)
  - p = large prime
  - (a, b) = coefficients in $y^2 = x^3 + ax + b$
  - G = base point that generates subgroup of large prime order
  - n = order of the subgroup
  - h = cofactor of the subgroup

ECDH – Elliptic Curve Diffie-Hellman

- Alice: $d_A \leftarrow_R \mathbb{Z}_n, \ \ H_A = d_A G$
- Bob: $d_B \leftarrow_R \ \mathbb{Z}_n, \ \ H_B = d_B G$

- Alice -> Bob: H_A
- Bob -> Alice: H_B

- Alice: $d_A H_B = d_A d_B G$
- Bob: $d_B H_A = d_B d_A G$

- Say $S = d_A d_B G$ is the shared secret, can use it to derive a symmetric key

ECDSA – Elliptic Curve Digital Signature Algorithm

- Public Information: (p, a, b, G, n, h)
- Alice's Private Key: d_A
- Alice's Public Key: $H_A = d_A G$

- Alice signs a message $m \in \mathbb{Z}_n$ by performing the following:
  - $k \leftarrow_R \mathbb{Z}_n$
  - $P = kG = (x_P, y_P)$
  - $r = x_P \ mod \ n$, if $r = 0$ start over
  - $s = k^{-1}(m + rd_A) \ mod \ n$, if $s = 0$ start over
  - Output signature (s, r)

ECDSA – Elliptic Curve Digital Signature Algorithm

- Bob can verify a message signed by performing the following:
  - Bob gets (m, s, r, H_A)
  - Calculate $u_1 = s^{-1}m \ \ mod\, n, \ u_2 = s^{-1}r \ \ mod\, n$
  - Calculate $P = u_1 G + u_2 H_A$
  - Valid if and only if $r = x_P\ mod\, n$

ECDSA – Elliptic Curve Digital Signature Algorithm

- Check that the algorithm is correct:
  - $P = u_1G + u_2H_A = u_1G + u_2d_AG = (u_1 + u_2d_A)G$
  - $P = (s^{-1}m + s^{-1}rd_A)G = s^{-1}(m + rd_A)G$
  - $s = k^{-1}(m + rd_A) \rightarrow k = s^{-1}(m + rd_A)$

  - $P = s^{-1}(m + rd_A)G = kG$ – Thus the signature will verify correctly

Acknowledgments

- Many slides created by Kyle Soska (TA for 18733 in Spring 2016)

Pairing Based Cryptography

- Computational Diffie-Hellman
  - Given g, g^a, g^b compute g^{ab}
- Decisional Diffie-Hellman
  - Given g, g^a, g^b, cant tell g^{ab} apart from random element g^c for random c

- Let G_1, G_2, G_T be groups of prime order q, then a bilinear pairing denoted e is an operation that maps from $G_1 \times G_2 \rightarrow G_T$ such that
- $\forall a, b \in \mathbb{F}_q, \forall P \in G_1, \forall Q \in G_2 \;\; e(aP, bQ) = e(P, Q)^{ab} \neq 1$

- Idea: We can use pairing based cryptography to create a situation where Computational Diffie-Hellman is hard, but Decisional Diffie-Hellman is easy

Pairing Based Cryptography

- Computational Diffie-Hellman
  - Given g, g^a, g^b compute g^{ab}
- Decisional Diffie-Hellman
  - Given g, g^a, g^b, cant tell g^{ab} apart from random element g^c for random c

- Suppose an adversary has g^a, g^b, g^z, where g^z is randomly either g^{ab} or g^c for c random. How can he check which one he has?
  - $e\left(g^a, g^b\right) = e(g,g)^{ab} = e\left(g, g^{ab}\right)$
  - Adversary computes $e(g, g^z) =?\, e\left(g^a, g^b\right)$

Pairing Based Signatures (Boneh et al.)

- $x \leftarrow_R \mathbb{Z}_q$
- Private Key: x, Public Key: g^x
- Sign message m by hashing it yielding $h = H(m)$ and signing the hash as $\sigma = h^x$
- Verify (σ, m) as $e(\sigma, g) =?\, e(H(m), g^x)$
  - $e(\sigma, g) = e(h^x, g) = e(H(m)^x, g) = e(H(m), g)^x = e(H(m), g^x)$

Twists Of Elliptic Curves

- Suppose you have an elliptic curve $E[p]$ over some field $\mathbb{F}$
- A twist of $E[p]$ another elliptic curve over a field extension of $\mathbb{F}$
- A twist of $E[p]$ will be isomorphic to $E[p]$, namely it will have the same order, and there is a 1-1 onto mapping between them

Other Notes

- Weil Pairing is a well studied paring where the groups G are elliptic curves
- There are many standardized elliptic curve groups
  - $y^2 + xy = x^3 + ax^2 + 1$ over $\mathbb{F}_{2^m}$, $m =$ prime and $a = 0$ or 1
    - Koblitz Curves, very fast addition and multiplication
  - $x^2 + y^2 = 1 + dx^2y^2$ where $d = 0$ or 1
    - Edwards Curves, point addition is the same in all cases, and reasonably fast