A New Method of Solving Numerical Equations of All Orders, by Continuous Approximation
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XXI. *A new method of solving numerical equations of all orders, by continuous approximation.** By W. G. Horner, Esq. Communicated by* Davies Gilbert, *Esq. F. R. S.*

Read July 1, 1819.

1. T H E process which it is the object of this Essay to establish, being nothing else than the leading theorem in the Calculus of Derivations, presented under a new aspect, may be regarded as a universal instrument of calculation, extending to the composition as well as analysis of functions of every kind. But it comes into most useful application in the numerical solution of equations.

2. ARBOGAST's developement of

$$\varphi \left(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \ldots \right)$$

(See *Calc. des Der.* § 33) supposes all the coefficients within the parenthesis to be known previously to the operation of φ. To the important cases in which the discovery of γ, δ, &c. depends on the previous developement of the partial functions

$$\varphi \left(\alpha + \beta x \right), \varphi \left(\alpha + \beta x + \gamma x^2 \right), \&c.$$

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* The only object proposed by the author in offering this Essay to the acceptance of the Royal Society, for admission into the Philosophical Transactions, is to secure beyond the hazard of controversy, an Englishman's property in a useful discovery. Useful he may certainly be allowed to call it, though the produce of a purely mathematical speculation ; for of all the investigations of pure mathematics, the subject of *approximation* is that which comes most directly, and most frequently into contact with the practical wants of the calculator.

How far the manner in which he has been so fortunate as to contemplate it has conduced, by the result, to satisfy those wants, it is not for him to determine ; but his belief is, that both Arithmetic and Algebra have received some degree of improvement, and a more intimate union. The abruptness of transition has been broken down into a gentle and uniform acclivity.

it is totally inapplicable. A theorem which should meet this deficiency, without sacrificing the great facilitating principle of attaching the functional symbols to a alone, does not appear to have engaged the attention of mathematicians, in any degree proportionate to the utility of the research. This desideratum it has been my object to supply. The train of considerations pursued is sufficiently simple ; and as they have been regulated by a particular regard to the genius of arithmetic, and have been carried to the utmost extent, the result seems to possess all the harmony and simplicity that can be desired ; and to unite to continuity and perfect accuracy, a degree of facility superior even to that of the best popular methods.

Investigation of the Method.

3. In the general equation

$$\varphi x = 0$$

I assume $x = R + r + r' + r'' + \ldots \ldots$
and preserve the binomial and continuous character of the operations, by making successively

$$x = R \ + z \ = R + r \ + z'$$
$$= R' \ + z' = R' + r' + z''$$
$$= R'' + z'' = \&c.$$

Where R^n represents the whole portion of x which has already been subjected to φ, and $z_x = r_x + z_{x'}$ the portion still excluded ; but of which the part r_x is immediately ready for use, and is to be transferred from the side of z to that of R, so as to change φR_x to $\varphi R_{x'}$ without suspending the corrective process.

4. By TAYLOR's theorem, expressed in the more convenient manner of ARBOGAST, we have

$$\varphi x = \varphi (R + z) =$$
$$\varphi R + D\varphi R . z + D^2 \varphi R . z^2 + D^3 \varphi R . z^3 + \ldots \ldots$$

Where by $D^n \varphi R$ is to be understood $\dfrac{d^n \varphi R}{1 . 2 \ldots . n . dR^n}$, viz. the n^{tb} derivee with its proper denominator; or, that function which ARBOGAST calls the *derivée divisée*, and distinguishes by a *c* subscribed. Having no occasion to refer to any other form of the derivative functions, I drop the distinctive symbol for the sake of convenience. Occasionally these derivees will be represented by $a, b, c,$ &c.

5. Supposing φR and its derivees to be known, the mode of valuing $\varphi R'$ or $\varphi (R + r)$ is obvious. We have only to say in the manner of LAGRANGE, when preparing to develope his Theory of Functions,

$$\varphi R' = \varphi R + Ar$$
$$A = D\varphi R + Br$$
$$B = D^2\varphi R + Cr$$
$$C = D^3\varphi R + Dr$$
$$\cdots \cdots \cdots$$
$$V = D^{n-2} \varphi R + Ur$$
$$U = D^{n-1} \varphi R + r \quad \cdots \cdots \cdots [\text{I.}]$$

Taking these operations in reverse order, we ascend with rapidity to the value of $\varphi (R + r)$ or $\varphi R'$.

6. The next point is, to apply a similar principle to discover the value of $\varphi (R + r + r') = \varphi (R' + r') = \varphi R''$. We here have

$$\varphi R'' = \varphi R' + A'r'$$
$$A' = D\varphi R' + B' r'$$
$$B' = D^2\varphi R' + C' r'$$
$$C' = D^3\varphi R' + D' r'$$
$$\cdots \cdots \cdots$$
$$V' = D^{n-2} \varphi R' + U' r'$$
$$U' = D^{n-1} \varphi R' + r'$$

But the former operation determined $\varphi R'$ only, without giving the value of any of the derived functions. The very simple

scale of known quantities, therefore, by which we advance so rapidly in the first process, fails in those which follow.

7. Still we can reduce these formulæ to known terms; for since we have in general

$$\mathrm{D}^r \, \mathrm{D}^s \, \varphi\alpha = \frac{r+1 \cdot r+2 \ldots\ldots\ldots r+s}{1 \quad 2 \quad\quad\quad s} \, \mathrm{D}^{r+s} \varphi\alpha$$

(See ARBOGAST, § 137); by applying a similar reduction to the successive terms in the developement of $\mathrm{D}^m \, \varphi \mathrm{R}' = \mathrm{D}^m \varphi(\mathrm{R} + r)$, we obtain*

$$\mathrm{D}^m\varphi\mathrm{R}' = \mathrm{D}^m\varphi\mathrm{R} + \frac{m+1}{1}\,\mathrm{D}^{m+1}\,\varphi\mathrm{R} \cdot r + \frac{m+1}{1} \cdot \frac{m+2}{2}\,\mathrm{D}^{m+2}\,\varphi\mathrm{R} \cdot r^2$$

$$+ \frac{m+1 \ldots\ldots m+3}{1 \quad\quad\quad 3}\,\mathrm{D}^{m+3}\varphi\mathrm{R} \cdot r^3 + \&\text{c.}$$

And it is manifest that this expression may be reduced to a form somewhat more simple, and at the same time be accommodated to our principle of successive derivation, by introducing the letters A, B, C, &c. instead of the functional expressions.

8. As a general example, let

$$\mathrm{M} = \mathrm{D}^m \quad \varphi\mathrm{R} + \mathrm{N}r$$
$$\mathrm{N} = \mathrm{D}^{m+1}\varphi\mathrm{R} + \mathrm{P}r$$
$$\mathrm{P} = \mathrm{D}^{m+2}\varphi\mathrm{R} + \mathrm{Q}r$$

. . . .

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* This theorem, of which that in Art. 4 is a particular case [$m = o$], has been long in use under a more or less restricted enunciation, in aid of the transformation of equations. HALLEY's *Speculum Analyticum*, NEWTON's limiting equations, and the formulæ in SIMPSON's Algebra (ed. 5, p. 166, *circa fin.*) are instances. In a form still more circumscribed [$r = 1$, $\mathrm{R} = o$, 1, 2, &c.] it constitutes the *Nouvelle Methode* of BUDAN; which has been deservedly characterized by LAGRANGE as simple and elegant. To a purpose which will be noticed hereafter, it applies very happily; but regarded as an instrument of approximation, its extremely slow operation renders it perfectly nugatory: and as LEGENDRE justly reported, and these remarks prove, it has not the merit of originality.

represent any successive steps in the series in Art. 5; then are

$$D^m \quad \varphi R = M - Nr$$
$$D^{m+1} \varphi R = N - Pr$$
$$D^{m+2} \varphi R = P - Qr$$

. . . .

And by substituting these equivalents in the developement just enounced, it becomes

$$D^m \varphi R' = M + m\,Nr + \frac{m\,.\,m+1}{1\quad 2}Pr^2 + \frac{m\,...\,m+2}{1\,...\,3}Qr^3 + \&c.$$

9. With this advantage, we may now return to the process of Art. 6, which becomes.

$$\varphi R'' = \varphi R' + A'\,r'$$
$$A' = (A + Br + Cr^2 + Dr^3 + Er^4 + \&c.) + B'\,r'$$
$$B' = (B + 2\,Cr + 3\,Dr^2 + 4\,Er^3 + \&c.) + C'\,r'$$
$$C' = (C + 3\,Dr + 6\,Er^2 + \&c.) + D'\,r'$$

.   .   .   .

$$V' = \left(V + \frac{n-2}{1}Ur + \frac{n-2\,.\,n-1}{1\quad.\quad2}r^2\right) + U'\,r'$$
$$U' = \left(U + \frac{n-1}{1}r\right) + r' \quad . \quad . \quad . \quad . \quad [\text{II.}]$$

Taking these operations in reverse order as before, by determining U', V' . . . . C', B', A', we ascend to the value of $\varphi R''$.

10. In this theorem, the principle of successive derivation already discovers all its efficacy; for it is obvious that the next functions U'', V'' . . . . . C'', B'', A'', $\varphi R'''$, flow from the substitution of A', B', C', ... V', U', $\varphi R''$, r', r'', for A, B, C . . . . V, U, $\varphi R'$, r, r', in these formulæ; and from these U''', V''', &c.; and so on to any desirable extent. In this respect, Theorem II, algebraically considered, perfectly answers the end proposed in Art. 2.

11. We perceive also, that some advance has been made toward arithmetical facility; for all the figurate coefficients

here employed are lower by one order than those which naturally occur in transforming an equation of the nth degree. But it is much to be wished, that these coefficients could be entirely dispensed with. Were this object effected, no multipliers would remain, except the successive corrections of the root, and the operations would thus arrange themselves, in point of simplicity, in the same class as those of division and the square root.

12. Nor will this end appear unattainable, if we recur to the known properties of figurate numbers; which present to our view, as *equivalent to the nth term of the mth series :*

1. *The difference of the nth and $n - 1^{th}$ term of the $m + 1^{th}$ series.*

2. *The sum of the first n terms of the $m-1^{th}$ series.*

3. *The sum of the nth term of the $m-1^{th}$, and the $n-1^{th}$ term of the mth series.*

The depression already attained has resulted from the first of these properties, and a slight effort of reflection will convince us that the second may immediately be called to our aid.

13. For this purpose, let the results of Art. 9 be expressed by the following notation :

$$\varphi R'' = \varphi R' + A' r'$$
$$A' = A_1 + B' r'$$
$$B' = B_2 + C' r'$$
$$C' = C_3 + D' r'$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad ,$$
$$V' = V_{n-2} + U' r'$$
$$U' = U_{n-1} + r'$$

the exponents subjoined to any letter indicating the degree

of the figurate coefficients in that formula of the theorem, of which such letter is the first term.

14. Although this statement appears only to have returned to us the conditions of Art. 6, with all their disadvantages, and to have merely substituted

$$A_1 \text{ for } D\varphi R' \text{ or } a'$$
$$B_2 \text{ for } D^2\varphi R' \text{ or } b'$$
$$C_3 \text{ for } D^3\varphi R' \text{ or } c'$$

&c. yet, by means of the property just alluded to, the essential data A, B, C, &c. which have disappeared, will again be extricated. For the developement of $D^m\varphi R'$, found in Art. 8, undergoes thereby the following analysis :

$$M + m\,Nr + \frac{m \cdot m+1}{1 \cdot 2} Pr^2 + \frac{m \cdot m+1 \cdot m+2}{1 \cdot 2 \cdot 3} Qr^3 + \ldots$$
$$= M + Nr + Pr^2 + Qr^3 + \ldots$$
$$+ Nr + 2\,Pr^2 + 3\,Qr^3 + \ldots$$
$$+ Nr + 3\,Pr^2 + 6\,Qr^3 + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$+ Nr + m\,Pr^3 + \frac{m \cdot m+1}{1 \cdot 2} Qr^3 + \ldots$$

which equivalence will be thus expressed:

$$M_m = M + N_1 r + N_2 r + N_3 r + \ldots \ldots + N_m r$$

Returning therefore once more to our theorem, we now have

$$\varphi R'' = \varphi R' + A'r'$$
$$A = (A + B_1 r) + B' r'$$
$$B' = (B + C_1 r + C_2 r) + C' r'$$
$$C' = (C + D_1 r + D_2 r + D_3 r) + D' r'$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$V' = (V + U_1 r + U_2 r + U_3 r + \ldots \ldots U_{n-1} r) + U' r'$$
$$U' = (U + \overline{n-1} \cdot r) + r'$$

15. This theorem employs exactly the same total number of addends as Theorem II, but with the important improvement, that the number of addends to each derivee is inversely as their magnitude, contrary to what happened before. Figurate multipliers are also excluded. And it is easy to convince ourselves that no embarrassment will arise from the newly introduced functions. For if we expand any of the addends $N_k r$ in the general formula equivalent to M_m, and analyze it by means of the *third property* of figurate series, we shall find

$$M_k r = N_{k-1} r + P_k rr.$$

And since we take the scale in our Theorem in a reverse or ascending order, this formula merely instructs us to multiply an addend already determined by r, and to add the product to another known addend; and if we trace its effect through all the descending scale, to the first operations, we observe that the addends to the last derivee, from which the work begins, are simply r repeated $n-1$ times.

16. Because $N_0 = N$, the addend exterior to the parenthesis, might for the sake of uniformity be written $N_0' r'$. The harmony of the whole scheme would then be more completely displayed. To render the simplicity of it equally perfect, we may reflect that as the factors r, r', &c. are engaged in no other manner than has just been stated, viz. in effecting the subordinate derivations, their appearance among the principal ones is superfluous, and tends to create embarrassment. Assume therefore

$$_k N = N_k r,$$

and we have.

$$\varphi R'' = \varphi R' + {}_{o}A'$$
$$A' = (A + {}_{1}B) + {}_{o}B'$$
$$B' = (B + {}_{1}C + {}_{2}C) + {}_{o}C'$$
$$C' = (C + {}_{1}D + {}_{2}D + {}_{3}D) + {}_{o}D'$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$V' = (V + {}_{1}U + {}_{2}U + {}_{3}U + \cdots \cdot {}_{n-2}U) + {}_{o}U'$$
$$U' = (U + \overline{n-1} \cdot r) + r' \qquad \cdot \qquad \cdot \qquad \cdot \qquad [\text{III}]$$

the subordinate derivations being understood.

17. The Theorems hitherto give only the synthesis of φx, when $x = R + r + r' + $ &c. is known. To adapt them to the inverse or analytical process, we have only to subtract each side of the first equation from the value of φx; then assuming $\varphi x - \varphi R^{x} = \Delta^{x}$, we have

$$\Delta' = \Delta - {}_{o}A$$
$$A = a + {}_{o}B$$

&c. as in Theorem I.

$$\Delta'' = \Delta' - {}_{o}A'$$
$$A' = (A + {}_{1}B) + {}_{o}B'$$

&c. as in Theorem II. or III.

The successive invention of R, r, r', &c. will be explained among the numerical details. In the mean time, let it be observed that these results equally apply to the popular formula $\varphi x = $ constant, as to $\varphi x = o$.

18. I shall close this investigation, by exhibiting the whole chain of derivation in a tabular form. The calculator will then perceive, that the algebraic composition of the addends no longer requires his attention. He is at liberty to regard the characters by which they are represented, in the light of mere corresponding symbols, whose origin is fully explained

at their first occurrence in the table, and their ultimate application at the second. The operations included in the parentheses may be mentally effected, whenever r is a simple digit. And lastly, the vertical arrangement of the addends adapts them at once to the purposes of arithmetic, on every scale of notation.

General Synopsis.

$n-1^{th}$ Der.	$n-2^{th}$ Derivee.	$n-3^{th}$ Derivee.	. . . .	3rd. Derivee.	2nd. Derivee.	1st. Derivee.	Synthesis.	Anal.	Root.
u	v	t		c	b	a	φR	Δ	$R+r+r'+$
r	$(Ur=)\ U_0$	$(rV=)\ V_0$	&c.	$Dr=)\ D_0$	$(Cr=)\ C_0$	$(Br=)\ B_0$	$(Ar=)\ A_0$	$-A_0$	
U	V	T	&c.	C	B	A	φR	Δ'	
$\overline{n-1}\,.\,r$	$(U+r^2=)\ U_1$	$(V+Ur=)\ V_1$		$(D+Er=)\ D_1$	$(C+Dr=)\ C_1$	$(B+Cr=)\ B'$	$(A'r'=)\ A'_0$	$-A'_0$	
	$(U+r^2=)\ U_2$	$(V+Ur=)\ V_2$		$(D+Er=)\ D_2$	$(C+Dr=)\ C_2$	$(B'r'=)\ B'_0$			
	&c. to U^{n-2}	&c. to V^{n-3}	&c.	$(D+Er=)\ D_3$	$(C'r'=)\ C'_0$		$\varphi R''$	Δ''	
r'	$(U'r'=)\ U'_0$	$(V'r'=)\ V'_0$		$(D'r'=)\ D'_0$		$(B'+C'r'=)\ B'_1$	$(A''r''=)\ A''_0$	$-A'''_0$	
U'	V'	T'		C'	C		&c.	&c.	
$\overline{n-1}\,.\,r'$	$(U'+r'^2=)\ U'_1$	$(V+U'r'=)\ V'_1$	&c.	$(D'+E'r'=)D'_0$	$(C'+D'r'=)C'_1$	B'			
&c.	&c.	&c.		&c.	&c	&c.			

Illustrations.

19. The remarks which are yet to be adduced will bear almost exclusively on the Analytic portion of the Theorem, from which the Synthetic differs only in the less intricate management of the first derivee; this function having no concern with the discovery of the root, and its multiple being additive like all the rest, instead of subtractive.

From the unrestricted nature of the notation employed, it is evident that no class of equations, whether finite, irrational or transcendental, is excluded from our design. In this respect indeed, the new method agrees with the established

popular methods of approximation; a circumstance in favour
of the latter, which is overlooked by many algebraists, both
in employing those methods, and in comparing them with
processes pretending to superior accuracy. The radical fea-
ture which distinguishes them from ours is this: they forego
the influence of all the derivees, excepting the first and
perhaps the second; ours provides for the effectual action
of all.

20. Concerning these *derivées* little need be said, as their
nature and properties are well known. It is sufficient to state
that they may be contemplated either as differential coeffi-
cients, as the limiting equations of NEWTON, or as the nume-
rical coefficients of the transformed equation in $R + z$. This
last elementary view will suffice for determining them, in
most of the cases to which the popular solutions are adequate;
viz. in finite equations where R, an unambiguous limiting
value of x, is readily to be conjectured. When perplexity
arises in consequence of some roots being imaginary, or dif-
fering by small quantities, the second notation must be called
in aid. The first, in general, when φx is irrational or trans-
cendental.

21. The fact just stated, namely, that our theorem con-
tains within itself the requisite conditions for investigating the
limits, or presumptive impossibility, of the roots, demonstrates
its sufficiency for effecting the developement of the real
roots, independently of any previous knowledge of R. For this
purpose, we might assume $R = o$; r, r, &c. $= 1$ or $.1$ &c. and
adopt, as most suitable to these conditions, the algorithm of
Theorem II, until we had arrived at R^x, an unambiguous
limiting value of x. But since these initiatory researches

seem more naturally to depend on the simple derivees, a, b, &c. than on A, B, &c. their aggregates ; and since, in fact, as long as r is assumptive or independent of R, our system of derivation offers no peculiar advantage ; I should prefer applying the limiting formulæ in the usual way ; passing however from column to column (Wood, § 318.) of the results, at first by means of the neat algorithm suggested in the note on Art. 7, and afterwards by differencing, &c. as recommended by La- grange, (*Res. des Eq. Num.* § 13), when the number of co- lumns has exceeded the dimensions of the equation. (Vide Addendum.)

If, during this process the observation of De Gua be kept in view, that whenever all the roots of φx are real, $D^{m-1} \varphi x$ and $D^{m+1} \varphi x$ will have contrary signs when $D^m \varphi x$ is made to vanish, we shall seldom be under the necessity of resorting to more recondite criteria of impossibility. Every column in which o appears between results affected with like signs, will apprize us of a distinct pair of imaginary roots; and even a horizontal change of signs, occurring between two horizontal permanences of an identical sign, will induce a suspicion, which it will in general be easy, in regard of the existing case, either to confirm or to overthrow.

22. The facilities here brought into a focus, constitute, I believe, a perfectly novel combination ; and which, on that account, as well as on account of its natural affinity to our own principles, and still more on account of the extreme degree of simplicity it confers on the practical investigation of limits, appears to merit the illustration of one or two familiar ex- amples.

Ex. 1. Has the equation $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$ any real root?—See EULER, C. D. p. 678.

$x =$	0	1	2
	20	9	4
	—16	—8	0
	8	2	8
	—4	0	4
	1	1	1

Here the first column consists of the given co-efficients taken in reverse order. In the second, 9 is = the sum of the first column, — 8 is = — 16 + 2 (8) + 3 (— 4) + 4 (1), 2 is = 8 + 3 (—4) + 6 (1), &c. The third column is formed from the second, by the same easy process. We need proceed no farther; for the sequences 2, 0, 1 in the second column, and 4, 0, 8 in the third, show that the equation has two pairs of imaginary roots. Consequently it has no real root.

Ex. 2. To determine the nearest distinct limits of the positive roots of $x^3 - 7x + 7 = 0$. See LAGRANGE, *Res. des E. N.* § 27, and note 4. § 8.

Operating as in the former example, we have

$x =$	0	1	2
	7	1	1
	—7	—4	5
	0	3	6
	1	1	1

Since all the signs are now positive, 2 is greater than any of the positive roots. Again, between — 4 and + 5, it is manifest, that *o* will occur as a value of the first derivee, and

that the simultaneous value of the second derivee will be affirmative. But as the principal result has evidently converged and subsequently diverged again in this interval, no conclusion relative to the simultaneous sign of that result can be immediately drawn. We will return to complete the transformations.

For $x=$ 1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
1000	631	328	97	—56	—125	—104	13
— 400	—337	—268					
30	33	36					
1	1	1					

Here the first column was formed from that under $x=1$, by annexing ciphers according to the dimensions of the functions; the 2nd and 3rd columns and the number 97 were found as in the former Example; the remaining numbers by differencing and extending the series 1000,631,328,97. We have no need to continue the work, since the changes of signs in the principal results indicate the first digits of the roots in question to be 1.3 and 1.6. But if we proceed by farther differencing to complete all the lines, the columns standing under these numbers will give the co-efficients of $\varphi(1.3 + z)$ and $\varphi(1.6 + z)$ without farther trouble.

23. Assuming, then, that R has been determined, and R $+$ z substituted for x in the proposed equation, thereby transforming it to

$$\Delta = az + bz^2 + cz^3 + dz^4 + \ldots\ldots\ldots$$

it is to this latter equation that the analytical part of our theorem is more immediately adapted. Now the slightest degree of reflection will evince, that our method is absolutely identical for all equations of the same order, whether they

be binomial or adfected, as soon as the transformation in R has been accomplished. The following description, therefore, of a familiar process in arithmetic, will convey an accurate general idea of our more extensive calculus, and obviate the necessity of any formal precepts.

In EVOLUTION, the first step is unique, and if not assisted by an effort of memory, could only be tentative. The whole subsequent process may be defined, *division by a variable divisor*. For an accurate illustration of this idea, as discoverable in the existing practice of arithmeticians, we cannot however refer to the mode of extracting any root, except that of the square; and to this, only in its most recently improved state. Here, in passing from one divisor to another, *two additive corrections* are introduced; the first depending on the last correction of the root, the second on the correction actually making. And this *new quotient correction* of the root, since it must exist previously to the completion of the divisor by which it is to be verified, *is required to be found by means of the incomplete divisor*; and may be taken out, either to one digit only, as is most usual, or to a number of digits equal to that which the complete and incomplete divisors possess in common. And farther, as these *divisors* may not, *in the first* instance, agree accurately even in a single digit, it is necessary at that stage of the operation, mentally to anticipate the effect of the new quotient, so as to obtain a sufficiently correct idea of the magnitude of the new divisor.
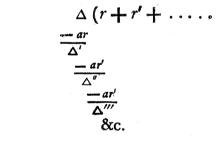
24. This is an accurate statement of the relation which the column headed by the first derivee bears to the analysis. The remaining columns contribute their aid, as successively subsidiary to each other; the contributions commencing with

the last or $n-1^{th}$ derivee, and being conveyed to the first through a regular system of *preparatory addends* dependent on the last quotient-correction, and of *closing addends* dependent on the new one. The *overt and registered* manner of conducting the whole calculation, enables us to derive important advantage from *anticipated corrections* of the divisors, not only at the first step, but, if requisite, through the whole performance, and also, without the necessity of a minute's bye-calculation, communicates, with the result, its *verification*.

25. Let us trace the operation of the theorem as far as may be requisite, through the ascending scale of equations.

1. In *Simple equations*, the reduced equation may be represented by $\Delta = az$; whence $z = \frac{\Delta}{a}$. Now the theorem directs us to proceed thus:

$$a \qquad\qquad \Delta\,(r + r' + \ldots\ldots$$
$$\frac{-\,ar}{\Delta'}$$
$$\frac{-\,ar'}{\Delta''}$$
$$\frac{-\,ar'}{\Delta'''}$$
$$\&\text{c.}$$

precisely the common arithmetical process of division.

2. In *Quadratics*, we have $\Delta = az + z^2$, and proceed in this manner :

$$1 \qquad a \qquad\qquad \Delta\,(r + r' + \ldots\ldots$$
$$\frac{r}{A} \qquad\qquad \frac{-Ar}{\Delta'}$$
$$r \qquad\qquad \frac{-A'\,r'}{\Delta''}$$
$$\frac{r'}{A'} \qquad\qquad \&\text{c.}$$
$$\&\text{c.}$$

the known arithmetical process for extracting the square root.

3. At *Cubic equations*, the aberration of the old practice of evolution commences, and our theorem places us at once on new ground.   We have here

$$\Delta = az + bz^2 + z^3$$

and must proceed thus :

$$
\begin{array}{cccc}
1 & b & a & \Delta(r+r+\cdots \\[4pt]
 & \dfrac{r}{B} & Br=_0B & \dfrac{-Ar}{\Delta'} \\[6pt]
 & 2r & \dfrac{}{A} & \\[4pt]
 & \quad r' & {}_0B+r^2=_1B & \dfrac{-A'r'}{\Delta''} \\[6pt]
 & \dfrac{}{B'} & \dfrac{B'r'}{A'} & \text{\&c.} \\[4pt]
 & \text{\&c.} & \text{\&c.} &
\end{array}
$$

This *ought to be* the arithmetical practice of the cube root, as an example will prove.

Ex. I. *Extract the cube root of* 48228544.

Having distributed the number into tridigital periods as usual, we immediately perceive that the first figure of the root is 3 = R. Consequently, the first subtrahend is $R^3=27$, the first derivee $3R^2=27$, the second $3R=9$; the third (=1,) need not be written.   Hence

$$48228544(364$$

$$
\begin{array}{lll}
9\cdot & 27\cdots & 27 \\
6 & \text{---}576 & \\
\hline
 & & 21228 \\
96 & 3276\text{---------}19656 \\
12\cdot & 612\cdots & \\
4 & \text{---}4336 & 1572544 \\
\hline
 & & \text{---}1572544 \\
1084 & 393136 &
\end{array}
$$

In this example the reader will perceive that no supplementary operations are concealed.   The work before him is complete, and may be verified mentally.   I need not intimate

how much more concise it is *than even the abbreviated statement of the old process.* (See Hutton's *Course.*)

The station of 1, 2, &c. numeral places respectively, which the closing addends occupy in advance of the preparatory ones, is an obvious consequence of combining the numeral relation of the successive root-figures with the potential relation of the successive derivees. In fact, as is usual in arithmetic, we tacitly regard the last root-figure as units, and the new one as a decimal fraction; then the common rules of decimal addition and multiplication regulate the vertical alineation of the addends.

26. The advantage of mental verification is common to the solution of equations of every order, provided the successive corrections of the root be simple digits: for the parenthetic derivations will, in that case, consist of multiplying a given number by a digit, and adding the successive digital products to the corresponding digits of another given number; all which may readily be done without writing a figure intermediate to these given numbers and the combined result. For this reason the procedure by single digits appears generally preferable.

Nevertheless, to assist the reader in forming his own option, and at the same time to institute a comparison with known methods on their own grounds, I introduce one example illustrative of the advantage which arises from the anticipatory correction of the divisors spoken of in Art. 24, when the object is to secure a high degree of convergency by as few approximations as possible. The example is that by which Newton elucidates his method. I premise as the depreciators of Newton do, that it is an extremely easy

problem ; and I say this to invite comparison, not so much with his mode of treating it, as with theirs.

Ex. II. What is the value of x in the equation $x^3 - 2x = 5$.

The root is manifestly a very little greater than 2. Make it $x = 2 + z$, and the equation becomes

$$1 = 10z + 6z^2 + z^3.$$

Hence, arranging the derivees,

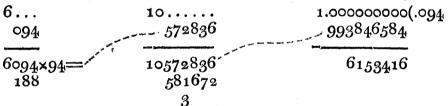<div align="center">6.   10 ..   1.000 (
6</div>

The first digit will obviously be so nearly 1, that by antici-pating its effect on the divisor, we are sure this will be very nearly 106. Hence

<div align="center">10.6) 1.000(.094 first correction</div>

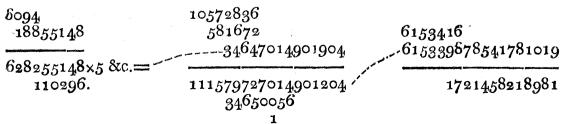The square is $94^2 = 8836$.

Hence we have

6...	10......	1.000000000(.094
094	572836	993846584
6094×94=	10572836	6153416
188	581672	
	3	

The first digit of the next correction will evidently be 5 ; the effect of which we have as before anticipated as far as one digit. The divisor will therefore be 11158 correct to the last figure. Hence

<div align="center">11158)6153416(55148, second correction.</div>

The square is 30413, &c. to 10 digits.

Hence,

6094	10572836	6153416
18855148	581672	6153398785417810 19
	34647014901904	
628255148×5 &c.=	1115797270149012 04	1721458218981
110296.	34650056	

1

Consequently,

$$1116143772)1721458218979(1542326590,22$$

This third correction is carried two places beyond the extent of the divisor, for the sake of ascertaining rigidly the degree of accuracy now attained. For this purpose, we proceed thus:

628 &c. × 154 &c. =, 968, &c. is the true correction of the last divisor. Our anticipated correction was 1,000. For which if we substitute 968 &c. it will appear that our divisor should have ended in 1,678, &c. instead of 2. The error is, ,322 &c. which induces an ultimate error of (111 &c. : 154 &c. : :, 322 &c. &c. :),44 &c. Consequently, our third correction should be . . . . 1542326590,66, &c. agreeing to 10 figures with the value previously determined. And the root is

$$x = 2.094551481542326590, \text{ &c.}$$

correct in the 18th decimal place at three approximations.

So rapid an advance is to be expected only under very favorable data. Yet this example clearly affixes to the new method, a character of unusual boldness and certainty; advantages derived from the overt manner of conducting the work, which thus contains its own proof.

The abbreviations used in the close of this example, are of a description sufficiently obvious and inartificial; but in order to perfect the algorithm of our method in its application to higher equations, and to the progress by simple digits, attention must be given to the following general principles of

Compendious Operation.

27. We have seen that every new digit of the root occasions the resolvend to be extended n figures to the right, and the mth derivee $n—m$ figures; so that if the work be carried on as with a view to unlimited progress, every new

root-figure will be obtained and verified at an expence of $\frac{1}{2}(n.n+3)+2$ new lines of calculation, containing in all somewhat above $\frac{1}{6}(n.n-1.n+4)$ digits more than the preceding root-figure cost. But as the necessity for unlimited continuity can rarely, if ever, occur, we may consider ourselves at liberty to check the advance of the resolvend, as soon as it contains one or two figures more than the number we yet propose to annex to the root. This will happen, generally speaking, when $\frac{1}{n}$ th of the numeral places of the root are determined.

By arresting the advance of the resolvend, we diminish it in the first instance by an optional number (p) of places, and by n places more at each succeeding step. Neglecting at the same time an equal number of figures in the right hand places of each closing addend and its derivatives, as contributing nothing to the diminished resolvend, we thus cause the effective units' place of each derivee to retrograde* in the first instance $p+m-n$ places, and at every subsequent step, a number of places (m) equal to the index of the derivee.

In the mean time, while these amputations are diminishing the derivees and addends on the right hand, a uniform average diminution of one digit on the left hand is taking place in the successive classes of addends in each column. The obvious consequence is, that after about $\frac{1}{m+1}$ th of the root-

---

* As the advance of the closing addend is prepared by annexing dots to the superior preparatory addend, so its retrogradation may be prepared by a perpendicular line beginning before the proper place (the mth or $m+p-n^{th}$) of the said preparatory addend, and continued indefinitely downward. One digit, or two, of those which fall to the right of this line in the next succeeding sum and preparatory addends, must be retained, for the sake ultimately of correctly adjusting the effective units of the subtrahend.

figures are found, the mth derivee will receive no augmenta-
tion; or, in other words, it will be exterminated when
$\frac{1}{m}$ th of those places are determined.

Again, when all the derivees inferior to M, the mth, have
vanished, the process reduces itself to that of the mth order
simply. For, the amount of the preparatory addends to L,
the $m—1^{th}$ augmented derivee, will be $m—1$ times the pre-
vious closing addend $_0$M; and the preparatory addends to
K, the $m—2^{th}$, will be formed from its previous closing addend
$_0$L, by adding $_0$Mr to it $m—2$ times successively; a proce-
dure obviously similar to that with which the general synopsis
commences.

28. From these principles we form the following conclu-
sions, demonstrative of the facilities introduced by this im-
provement on the original process:

1. Whatever be the dimensions (n) of the proposed equa-
tion, whose root is to be determined to a certain number of
places, only $\frac{1}{n}$ th part of that number (reckoning from the
point at which the highest place of the closing addend begins
to advance to the right of that of the first derivee) needs to
be found by means of the process peculiar to the complete
order of the equation; after which, $\frac{1}{n \cdot n—1}$ may be found by
the process of the $n—1^{th}$ order, $\frac{1}{n—1 \cdot n—2}$ by that of the $n—2^{th}$
order, &c.

2. Several of these inferior processes will often be passed
over *per saltum;* and when this advantage ceases, or does not
occur, the higher the order of the process, the fewer will be
the places determinable by it. And in every case, the latter

half of the root will be found by division simply. Meantime, the number of figures employed in verification of each successive root-digit, instead of increasing, is rapidly diminishing.

3. The process with which we commence, need not be of a higher order than is indicated by the number of places to which we would extend the root; and may be even reduced to an order as much lower as we please, by means of an introductory approximation.

Ex. III. Let the root of the equation in Ex. II. be determined to the tenth place of decimals.

Arranging the derivees as before, we proceed thus:

Consequently the root is 2.09455148.15, correct to the proposed extent, as appears on comparing it with the more enlarged value already found. The work occupied a very few minutes, and may be verified by mere perusal, as not a figure was written besides those which appear. By a similar operation, in less than half an hour, I have verified the root to the whole extent found in Ex. II.

Ex. IV. As a praxis in case of the intervention of negative numbers, let it be proposed to extract, to a convenient extent, that root of the equation $x^3 - 7x = -7$, whose first digits we have already determined to be 1.3. (See Art. 22.)

Making $x = 1.3 + z$, we have

$$-.097 = -1.93\,z + 3.9\,z^2 + z^3$$

Wherefore

```
 39·             −1 9 3 · ·              −97(.056895867
   5              − 1 9 7 5              −86625
 ─────          ─────────               ─────────
 395             −1 7 3 4 5 · · ·        −10375000
 106              2 0 0 0 · ·            −9048984
 ─────           ─ − 2 4 3 3 6          ─────────
 40 56            ───────────           −1326016
    12·          −1 5 0 8 1 6 4         −1184430
 ──────           2 4 3 7 2 · ·         ─────────
1··40 68          3 2 5 4 4 ·           −14586
                 ─────────              −132923
                 −1 4 8 0 5 3 76        ─────────
                   3 2 5 50             −8663
                     3 6 6              −7383
                 ───────────            ─────────
                 −1 4 7 7 9 1 7         −1280
                     3 6 6              −1181
                       2               ─────────
                 −1 4 7 6 5 3           −99
                                        −89
                                       ─────────
                                        −10
                                        −10
```

Consequently the root is 1.356895867. This example was selected by LAGRANGE for its difficulty.

Of its three roots, that which we have now found is the most difficult to obtain; yet the whole work, including the preparatory portion in Art. 22, may be performed without one subsidiary figure, in less than a quarter of an hour.

A little attention and practice will render the mental aggregation of positive and negative numbers as familiar as the addition of either sort separately. The introduction of a small negative resolvend, instead of a large positive one, where the opportunity occurs, will then greatly abridge the operations. For example, the cube root of 2, or 1.26..

$$\frac{-\ 0078950105}{1\ .\ 259921049895}$$

was determined to this extent, true to the 12th decimal place, within as small a compass and as short a time as the result of Example III.

Ex. V. As an example of a finite equation of a higher order, let the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 321$ be proposed. The root appears to be $> 2, < 3$; and the equation in $z = x - 2$ is

$$207 = 201z + 150z^2 + 59z^3 + 12z^4 + z^5$$

Hence,

59 . .	150 . . .	201	207.00000 /. 6386058033 27
756	39936	1139616	188 97696
6656	189936	3149616	18 0230400000
792	44688	1407744	13 9304119743
828	49656 . . .	861106581	40 926280257
864 . .	2755527	46434706581	38031030021
4509	287035527	869413824	2895250236
918509	2769081	234667126	2867504754
4518	2782662	475387875266	27745482
4527	746632	235265192	23904795
4536	293333902	176931184	3840687
120	747592	477917459068	3824781
93329	748550	176965668	15906
. . 360	664	1475	14343
94	2948864	478095900	1563
	. . 1128	1475	1434
	. 295	23	129
		478097617	96
		23	33
			33

Wherefore the root is 2.638605803327, correct to the 12th decimal, and capable, like the former results, of being verified by simple inspection. The other roots are imaginary; for when $- x = .4$, the fourth derivee vanishes between two affirmative results, and when $- x = .7$ &c., the second disappears under similar circumstances. (Arts. 21, 22.)

It appears to me, that no explanation of this solution can be offered, which has not been abundantly anticipated in this Essay; and the student who peruses it in connection with the General Synopsis, and Arts. 23, 27, will acquire an indelible impression of the whole algorithm.

Ex. VI. If it were proposed to obtain a very accurate solution of an equation of very high dimensions, or of the irrational or transcendental kind, a plan similar to the following might be adopted. Suppose, for example, the root of

$$x^x = 100, \text{ or } x \log x = 2$$

were required correct to 60 decimal places. By an easy experiment we find $x = 3.6$ nearly; and thence, by a process of the *third* order, $x = 3.597285$ more accurately.

Now, $3597286 = 98 \times 71 \times 47 \times 11$, whose logarithms, found to 61 decimals in SHARPE's Tables, give $R \log R = 2.00000096658$, &c. correct to 7 figures; whence the subsequent functions need be taken out to 55 figures only. They are

$$a = \text{Mod} + \log R = .990269449408, \text{ \&c.}$$
$$b = \text{Mod} \div 2R \quad = \ldots \ldots 0^760364, \text{ \&c.}$$
$$c = - b \div 3R \quad = \ldots \ldots -.0^{14}55, \text{ \&c.}$$

&c. The significant part disappears after the 8th derivee; consequently, the process will at first be of the *eighth* order. If the root is now made to advance by single digits, the first

of these will reduce the process to the *seventh* order; one more reduces it to the *sixth* order; two more, to the *fifth*, &c. The last 27 figures will be found by division alone.

But if the first additional correction is taken to 8 figures, and the second to 16, on the principle of Example II, we pass from the 8th order to the 4th at once, and thence to the 1st or mere division, which will give the remaining 29 figures. This mode appears in description to possess the greater simplicity, but is perhaps the more laborious.

It cannot fail to be observed, that in all these examples a great proportion of the whole labour of solution is expended on the comparatively small portion of the root, which is connected with the leading process. The toil attending this part of the solution, in examples similar in kind to the last, is very considerable; since every derivee is at this stage to receive its utmost digital extent. To obviate an unjust prejudice, I must therefore invite the reader's candid attention to the following particulars :

In all other methods the difficulty increases with the extent of the root, nearly through the whole work; in ours, it is in a great measure surmounted at the first step: in most others, there is a periodical recurrence to first conditions, under circumstances of accumulating inconvenience; in the new method, the given conditions affect the first derivees alone, and the remaining process is *arithmetically direct*, and increasingly easy to the end.

The question of practical facility may be decided by a very simple criterion; by comparing the *times* of calculation which I have specified, with a similar datum by Dr. HALLEY in favor of his own favorite method of approximation. (Philosophical Transactions for 1694.)

Addendum I. (*Vide* Art. 21.) *Note.* But in this case, it will be more elegant to find the differences at once by the theorem

$$\Delta^{t+1}D^m\varphi R'=\tfrac{m+1}{1}\Delta^t D^{m+1}\varphi R.r+\tfrac{m+1}{1}\cdot\tfrac{m+2}{2}\Delta^t D^{m+2}\varphi R.r^2+ \&c.$$

which, supposing r to be constant, is a sufficiently obvious corollary to the theorem in Art. 7. All the results may then be derived from the first column by addition. Thus, for the latter transformations in Ex. II. Art. 22, the preparatory operation would be

1st. Terms.	Diff. 1st.	2nd.	3rd.
1000	—369	66	6
—400	63	6	
30	3		
1			

and the succeeding terms would be found by adding these differences in the usual way to the respective first terms.

Addendum II. It is with pleasure that I refer to the Imperial Encyclopædia (Art. Arithmetic) for an improved method of extracting the cube root, which should have been noticed in the proper place, had I been aware of its existence; but it was pointed out to me, for the first time, by the discoverer, Mr. EXLEY, of Bristol, after this Essay was completed. It agrees in substance with the method deduced in Art. 25, from my general principle, and affords an additional illustration of the affinity between that principle and the most improved processes of common arithmetic.